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**R-148**

**THE CRITICAL INCLINATION PROBLEM IN  
SATELLITE ORBIT THEORY**

**By WILLIAM A. MERSMAN**

**1962**

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**By WILLIAM A. MERSMAN**

**Ames Research Center  
Moffett Field, Calif.**

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## THE CRITICAL INCLINATION PROBLEM IN SATELLITE ORBIT THEORY

By WILLIAM A. MERSMAN

### SUMMARY

*Solutions of the satellite orbit problem are obtained that do not exhibit singularities at the critical inclination angle. Series representations are obtained, their regions of convergence are exhibited, and quantitative measures of their speeds of convergence are provided for use in numerical computations.*

### INTRODUCTION

The first attempts to develop a theory of satellite orbits about an oblate planet (refs. 1, 2, 3) produced solutions containing a singularity at the "critical inclination angle," are  $\tan 2$ . The singularity occurs only in the long-period terms of the solution and can be removed if these terms are isolated and treated by special methods (refs. 4, 5, 6).

If only the "main problem" (ref. 1) is considered, in which the series for the planet's gravitational potential is truncated at the second harmonic, the principal results are that the apsidal motion is purely oscillatory when the inclination angle lies within a certain region (the libration region) centered at the critical value; pericenter oscillates about the node and never reaches the points of maximum declination. When the inclination angle lies outside the libration region, the apsidal motion is essentially secular with small oscillations superimposed (refs. 1, 2, 3). On the boundary between these two regions the apsidal motion is aperiodic, and pericenter approaches the point of maximum declination asymptotically.

Extension of the series representation of the gravitational potential to include the fourth harmonic merely shifts the stable position of pericenter, in the libration region, from the node to the point of maximum declination.

These well-known results, for the main problem, are obtained here by a new method, based on the intermediate orbit of reference 7. The use of the intermediate orbit rather than the osculating ellipse as a point of departure has certain attractive features that lead to mathematical simplification and improvement. The simplification consists of introducing a parameter that is a generalization of elliptical eccentricity, but which is a constant for the intermediate orbit. This eliminates troublesome apparent singularities for nearly circular orbits.

The improvement consists of presenting the solution in the form of infinite series, the first-term representing the previously published solutions, and of determining precisely the regions of convergence. Furthermore, precise measures of the speeds of convergence are obtained, so that truncation for numerical purposes can be effected with full error control.

### TABLE OF IMPORTANT SYMBOLS

$a$	}	parameters involving inclination angle, eccentricity, and oblateness parameter
$b$		
$x_0$		
$c$		$\cos I$
$e_0$		eccentricity
$E, K$		complete elliptic integrals
$I$		inclination angle
$j$		$\frac{J}{p^2}$
$J$		Jeffreys' oblateness constant
$J_n, I_n$		coefficients in series of elliptic functions
$k, k_1$		modulus of elliptic function
$O( )$		order of magnitude
$p$		semilatus rectum, dimensionless

$p_\rho, p_\varphi, p_\Omega$	conjugate momenta in canonical equations, dimensionless
$q$	perigee distance, dimensionless
$s$	$\sin I$
$t$	time, dimensionless
$v$	true anomaly
$\varphi$	argument of latitude
$\nu$	convergence parameter, measure of speed of convergence
$\rho$	geocentric distance, dimensionless
$\xi$	dimensionless variable measuring deviation of inclination angle or perigee distance from its initial value
$\xi_n$	coefficients in Fourier series for $\xi$
$\omega$	argument of pericenter
$\omega'$	$\frac{d\omega}{dv}$ , apsidal velocity
$\omega_n$	coefficients in Fourier series for $\omega$
$\Omega$	right ascension of ascending node

## SUBSCRIPTS

0	initial conditions
1	modulus $k_1$
$c$	values at convergence boundary
$s$	values at separatrix

## SUPERSSCRIPTS

$^{\circ}, ', ''$	degrees, minutes, seconds of arc
$\sim$	periodic component

## BINOMIAL COEFFICIENTS

$$\binom{j}{k} = \frac{j!}{k!(j-k)!}$$

## EQUATIONS OF MOTION

## CANONICAL COORDINATES

It has been shown in reference 7 that the problem of artificial satellite motion can be described in dimensionless, canonical form as follows. Let  $R$ , the equatorial radius of the planet, be taken as the unit of length, and  $\sqrt{R^3/GM}$  as the unit of time. Then the following dimensionless variables constitute a canonical set (see fig. 1):

$\rho$	radial distance
$\varphi$	argument of latitude
$\Omega$	right ascension of the ascending node
$p_\rho$	radial velocity
$p_\varphi$	angular momentum

$p_\Omega$	component of angular momentum along the planet's polar axis
$t$	time

Corresponding to equations (8) of reference 7, the canonical equations, in the absence of dissipative forces, are

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i} \quad (1)$$

where  $x_i$  represents any of the coordinates ( $\rho, \varphi, \Omega$ ) and  $y_i$  the conjugate momentum ( $p_\rho, p_\varphi, p_\Omega$ ). The Hamiltonian,  $H$ , is defined by

$$H = \frac{1}{2} p_\rho^2 + \frac{1}{2} \left( \frac{p_\varphi}{\rho} \right)^2 - \frac{1}{\rho} - S(\rho, \varphi, \Omega, I) \quad (2)$$

where  $S$  is the disturbance function, and the inclination angle,  $I$ , is given by

$$\cos I = \frac{p_\Omega}{p_\varphi} \quad (3)$$

For the main problem of artificial satellite theory the disturbance function is (eq. (10) of ref. 7)

$$S = \frac{J}{\rho^3} \left( \frac{1}{3} - \sin^2 I \sin^2 \varphi \right) = \frac{J}{\rho^3} \left( -\frac{1}{6} + \frac{1}{2} c^2 + \frac{1}{2} s^2 \cos 2\varphi \right) \quad (4)$$

where

$$c = \cos I \\ s = \sin I$$

and  $J$  is Jeffreys' constant for the oblateness. Its numerical value for Earth is 0.00162392 (ref. 8, p. 17, with  $J = 1.5J_2$ ).

## REDUCTION TO TWO DIMENSIONS

Whenever the disturbance function possesses axial symmetry, the problem can be reduced essentially to two dimensions as follows. Axial symmetry implies

$$\frac{\partial S}{\partial \Omega} = 0$$

Hence, the motion of the node is given by

$$\left. \begin{aligned} p_\Omega &= \text{constant} \\ \frac{d\Omega}{dt} &= \frac{1}{p_\varphi \sin I} \frac{\partial S}{\partial I} \end{aligned} \right\} \quad (5)$$

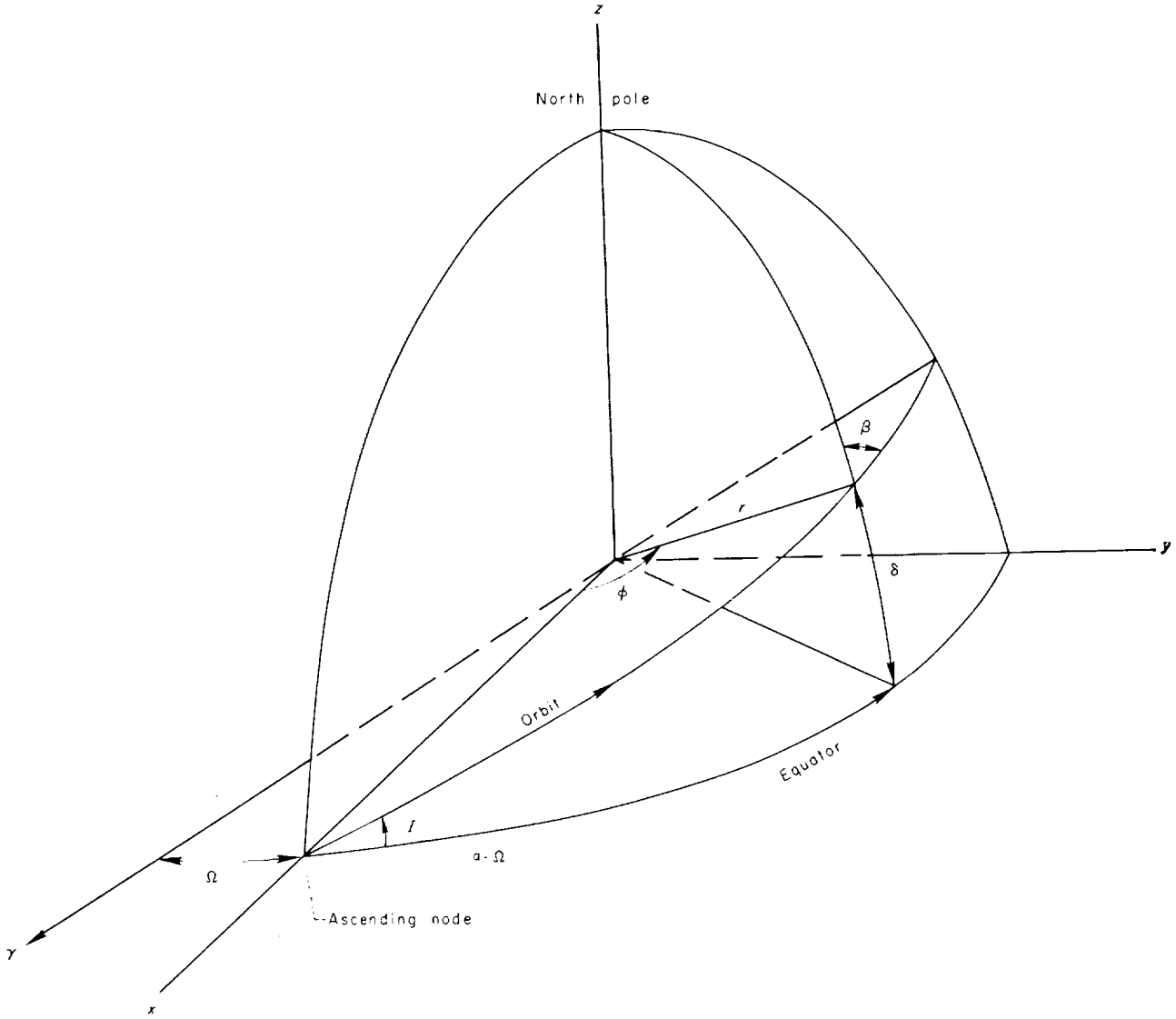


FIGURE 1.—Coordinate system.

The remaining canonical equations become

$$\left. \begin{aligned} \frac{d\rho}{dt} &= p_\rho, \quad \frac{dp_\rho}{dt} = \frac{p_\varphi^2}{\rho^3} - \frac{1}{\rho^2} + \frac{\partial S}{\partial \rho} \\ \frac{d\varphi}{dt} &= \frac{p_\varphi}{\rho^2} - \frac{\cot I}{p_\varphi} \frac{\partial S}{\partial I}, \quad \frac{dp_\varphi}{dt} = \frac{\partial S}{\partial \varphi} \end{aligned} \right\} \quad (6)$$

This set of equations is self-contained, since  $\Omega$  does not enter it, and  $p_\Omega$  is constant. Once it has been solved, the motion of the node can be obtained directly from equation (5) by quadrature.

It is also convenient to introduce a new independent variable in place of the time,  $t$ . Even in the classical, two-body problem such a transformation

is customary. Following the procedure of reference 7, define an auxiliary angle,  $\theta$ , by

$$\frac{d\theta}{dt} = \frac{d\varphi}{dt} + \frac{d\Omega}{dt} \cos I = \frac{p_\varphi}{\rho^2} \quad (7)$$

with the initial condition

$$p_\rho = 0 \text{ when } \theta = 0$$

The nodal equation now becomes

$$\frac{d\Omega}{d\theta} = \frac{\rho^2}{p_\varphi^2} \left( \frac{1}{\sin I} \right) \frac{\partial S}{\partial I} \quad (8)$$

The time equation becomes

$$\frac{dt}{d\theta} = \frac{\rho^2}{p_\phi} \quad (9)$$

and the two-dimensional problem becomes

$$\left. \begin{aligned} \frac{d\rho}{d\theta} &= \frac{\rho^2 p_\rho}{p_\phi}, \quad \frac{dp_\rho}{d\theta} = \frac{p_\phi}{\rho} - \frac{1}{p_\phi} + \frac{\rho^2}{p_\phi} \frac{\partial S}{\partial \rho} \\ \frac{d\phi}{d\theta} &= 1 - \frac{\rho^2 \cot I}{p_\phi^2} \frac{\partial S}{\partial I}, \quad \frac{dp_\phi}{d\theta} = \frac{\rho^2}{p_\phi} \frac{\partial S}{\partial \phi} \end{aligned} \right\} \quad (10)$$

Equations (8) and (9) for  $\Omega$  and  $t$  will not be considered further in this report. The essential nature of their solution has been adequately discussed in reference 7 (eqs. (64) and (72)).

#### INTRODUCTION OF TRUE ANOMALY AS INDEPENDENT VARIABLE

Introducing the disturbance function  $S$  from equation (4) gives the two-dimensional problem

$$\left. \begin{aligned} \frac{d\rho}{d\theta} &= \frac{\rho^2 p_\rho}{p_\phi} \\ \frac{dp_\rho}{d\theta} &= \frac{p_\phi}{\rho} - \frac{1}{p_\phi} + \frac{J}{p_\phi \rho^2} \left( \frac{1}{2} - \frac{3}{2} e^2 - \frac{3}{2} s^2 \cos 2\varphi \right) \\ \frac{d\varphi}{d\theta} &= 1 + J \frac{c^2}{p_\phi^2 \rho} (1 - \cos 2\varphi) \\ \frac{dp_\phi}{d\theta} &= -J \frac{s^2}{p_\phi \rho} \sin 2\varphi \end{aligned} \right\} \quad (11)$$

Now introduce the true anomaly,  $v$ , as independent variable, and the quantities  $U$ ,  $V$ ,  $\xi$ ,  $\omega$  as dependent variables, defined as follows:

$$\left. \begin{aligned} U &= \frac{p}{\rho} \\ V &= p_\rho p_\phi \\ \xi &= \frac{p_0}{p} - 1 = \frac{c^2}{c_0^2} - 1 = \frac{s_0^2 - s^2}{c_0^2} \\ \omega &= \varphi - v \end{aligned} \right\} \quad (12)$$

where

$$p = p_\phi^2$$

and the subscript 0 denotes initial values taken at any specific perigee. Note that the symbol  $\xi$  has a different meaning here than in reference 7. The true anomaly  $v$  will be regarded temporarily as an

arbitrary function of  $\theta$ , subject only to the initial conditions:

$$\left. \begin{aligned} \theta &= 0 \\ U &= 1 + e_0 \\ V &= 0 \\ \xi &= 0 \\ p &= p_0 \\ \omega &= \omega_0 \\ I &= I_0 \end{aligned} \right\} \text{when } v=0 \quad (13)$$

The definition of  $v$  will be completed in the next section.

This transformation is motivated by the fact that, when  $J=0$ , the solution is

$$\left. \begin{aligned} v &= \theta \\ U &= 1 + e_0 \cos v \\ V &= e_0 \sin v \\ \xi &= 0 \\ \omega &= \omega_0 \\ p &= p_0 \\ I &= I_0 \end{aligned} \right\} \text{when } J=0$$

Thus the constants  $e_0$ ,  $\omega_0$ ,  $p_0$ ,  $I_0$  are the usual osculating elliptical values of eccentricity, argument of pericenter, semilatus rectum, and inclination angle, evaluated at an arbitrary, specific pericenter chosen as the epoch.

The equations of motion now become

$$\left. \begin{aligned} \frac{d\xi}{dv} &= 2js^2U(1+\xi) \sin 2\varphi \frac{d\theta}{dv} \\ \frac{d\omega}{dv} &= -1 + [1 + jc^2U(1 - \cos 2\varphi)] \frac{d\theta}{dv} \\ \frac{dU}{dv} &= (-V - 2js^2U^2 \sin 2\varphi) \frac{d\theta}{dv} \\ \frac{dV}{dv} &= \left[ U - 1 + \frac{1}{2} jU^2(1 - 3c^2 - 3s^2 \cos 2\varphi) \right. \\ &\quad \left. - js^2VU \sin 2\varphi \right] \frac{d\theta}{dv} \end{aligned} \right\} \quad (14)$$

where

$$\left. \begin{aligned} j &= \frac{J}{p^2} = \frac{J}{p_0^2} (1+\xi)^2 \\ p &= \frac{p_0}{1+\xi} \\ c^2 &= c_0^2 (1+\xi) \\ s^2 &= s_0^2 - c_0^2 \xi \\ \varphi &= v + \omega \end{aligned} \right\} \quad (15)$$

An approximate solution of these equations, accurate to  $O(J^2)$ , will be obtained by an iterative procedure. At the first step, the terms containing  $\varphi$  explicitly in the equations for  $\xi$ ,  $U$ , and  $V$  will be neglected. The true anomaly,  $v$ , can then be defined in such a way that  $U$  and  $V$ , and hence  $\rho$  and  $d\rho/dt$ , contain no secular terms (see ref. 7).

This approximate solution for  $U$  can then be inserted in the complete equations for  $\xi$  and  $\omega$ , which can be solved by neglecting the short-period terms. The solution obtained in this way contains all the essential features of the orbit as regards the motion of pericenter, and constitutes the objective of the present report. In principle, the iterative procedure could be continued, by inserting the solution for  $\xi$  and  $\omega$  back into the complete equations for  $U$ ,  $V$ , etc. While this has not been done, it seems plausible a priori that the net effect of such iterations would be to introduce short-period terms which would have no essential effect on the qualitative nature of the motion.

#### THE INTERMEDIATE ORBIT

Following the procedure outlined in the preceding section, define the intermediate orbit as the solution of the simplified set of equations

$$\left. \begin{aligned} \frac{d\xi}{dv} &= 0 \\ \frac{dU}{dv} &= -V \frac{d\theta}{dv} \\ \frac{dV}{dv} &= (U-1-\sigma U^2) \frac{d\theta}{dv} \end{aligned} \right\} \quad (16)$$

where

$$\sigma = \frac{j}{2} (3c^2 - 1) \quad (17)$$

The first equation gives  $\xi=0$ , and hence,  $p$ ,  $s$ ,  $c$ ,  $\sigma$ , and  $I$  are constants for the intermediate orbit.

Equations (16) for  $U$  and  $V$  are equivalent to the first, fourth, and fifth of equations (12) of reference 7; that is, the intermediate orbit here is identical to that of reference 7. It will be rederived here by the following, simpler procedure, which is sufficient for the present restricted problem. Multiplying the  $V$  equation by  $2V$ , the  $U$  equation by  $2U$ , adding and integrating yields the energy integral

$$V^2 - 2U + U^2 - \frac{2}{3} \sigma U^3 = \eta = \text{constant} \quad (18)$$

where  $\eta$  is obtainable from the initial conditions:

$$\eta = e_0^2 - 1 - \frac{2}{3} \sigma (1 + e_0)^3 \quad (19)$$

It was shown in reference 7, equations (53), (54), (58), that the true anomaly,  $v$ , can be defined as a function of  $\theta$  by means of elliptic functions in such a way that  $\rho$  and  $d\rho/dt$  contain no secular terms. A more appropriate form of the solution for the present purpose, that avoids the use of elliptic functions, is:

$$\left. \begin{aligned} U &= u_0 + u_1 \cos v \\ \frac{d\theta}{dv} &= \frac{\sqrt{f}}{\sqrt{1 - k_1 U}} \\ V &= -\frac{dU}{dv} \frac{dr}{d\theta} = \frac{u_1}{\sqrt{f}} \sin v \sqrt{1 - k_1 U} \end{aligned} \right\} \quad (20)$$

where  $u_0$ ,  $u_1$ ,  $f$ ,  $k_1$  are constants, determined as follows: Insert equations (20) in the energy integral, rearrange as a cubic polynomial in  $U$ , and equate the coefficients to zero. The resulting equations can be transformed, by means of the initial condition

$$u_0 + u_1 = 1 + e_0 \quad (21)$$

into

$$\left. \begin{aligned} f &= 1 + \frac{4}{3} \sigma f^2 \left( 1 + \frac{1}{3} \sigma \eta f \right) \\ k_1 &= \frac{2}{3} \sigma f \\ u_0 &= f \left( 1 + \frac{1}{2} k_1 \eta \right) \\ u_1 &= 1 + e_0 - u_0 \end{aligned} \right\} \quad (22)$$

Since  $|\sigma| < |J| \ll 1$ , the equation for  $f$  can be solved by the obvious iterative procedure to any desired accuracy. Finally, if the radical  $\sqrt{1-k_1 U}$  is expanded by the binomial theorem and rearranged as a Fourier series, the solution can be written in the form

$$\left. \begin{aligned} U &= u_0 + u_1 \cos v \\ \theta &= v_0 v + \sum_{n=1}^{\infty} \frac{1}{n} v_n \sin nv \\ V &= \sum_{n=1}^{\infty} V_n \sin nv \end{aligned} \right\} \quad (23)$$

where, to  $O(\sigma^2)$ , the coefficients are

$$\left. \begin{aligned} u_0 &= 1 + \sigma \left( 1 + \frac{1}{3} e_0^2 \right) + \sigma^2 \left( 2 - \frac{2}{3} e_0 + \frac{2}{3} e_0^2 - \frac{2}{9} e_0^3 \right) \dots \\ u_1 &= e_0 - \sigma \left( 1 + \frac{1}{3} e_0^2 \right) - \sigma^2 \left( 2 - \frac{2}{3} e_0 + \frac{2}{3} e_0^2 - \frac{2}{9} e_0^3 \right) \dots \\ v_0 &= 1 + \sigma + \sigma^2 \left( \frac{5}{2} + \frac{5}{12} e_0^2 \right) \dots \\ v_1 &= \frac{1}{3} e_0 \sigma + \sigma^2 \left( -\frac{1}{3} + e_0 - \frac{1}{9} e_0^2 \right) \dots \\ v_2 &= \frac{1}{12} e_0^2 \sigma^2 \dots \\ v_n &= O(\sigma^n) \dots \\ V_1 &= e_0 - \sigma \left( 1 + e_0 + \frac{1}{3} e_0^2 \right) - \sigma^2 \left( 1 + \frac{5}{6} e_0 + \frac{1}{3} e_0^2 + \frac{1}{8} e_0^3 \right) \dots \\ V_2 &= -\frac{1}{6} e_0^2 \sigma + \sigma^2 \left( \frac{1}{3} e_0 - \frac{1}{6} e_0^2 + \frac{1}{9} e_0^3 \right) \dots \\ V_3 &= -\frac{1}{72} e_0^3 \sigma^2 \dots \\ V_n &= O(\sigma^{n-1}) \end{aligned} \right\} \quad (24)$$

It may be remarked that this procedure can be generalized. If higher harmonics are to be included in the planet's gravitational potential, it is merely necessary to add two powers of  $U$  to the

radicand (eq. (20)) for each even-ordered harmonic. Similarly, the secular effects of lunar and solar perturbations can be handled by adding negative powers of  $U$  to the radicand. In every case the number of constants ( $u_0, u_1, f$ , etc.) is one greater than the degree of the energy integral, considered as a polynomial in  $U$ , so that the procedure is determinate. The procedure of reference 7 cannot be generalized so easily.

While equation (23) for  $\theta$  in terms of  $v$  could be inverted to exhibit  $v$  explicitly as a function of  $\theta$ , such a procedure could serve no useful purpose here. All that is needed is  $d\theta/dv$  as a function of  $v$ , since this is the quantity that occurs in all the differential equations. Ultimately every quantity of physical interest can be expressed explicitly as a function of  $v$ , so that it can be regarded as a prime independent variable whose physical significance is of no real concern.

#### THE APSIDAL EQUATIONS

If the solution for the intermediate orbit is substituted in the general equations (14), the equations for  $\xi$  and  $\omega$  become, to  $O(J^2)$ :

$$\begin{aligned} \frac{d\xi}{dv} &= 2j_0 \left\{ s_0^2 \left[ 1 + \sigma_0 \left( 2 + \frac{1}{2} e_0^2 \right) \right] \right. \\ &\quad \left. + (4s_0^2 - 1)\xi \right\} \sin(2v + 2\omega) \\ &\quad + j_0 \left\{ s_0^2 \left[ e_0 + \sigma_0 \left( -1 + \frac{4}{3} e_0 - \frac{1}{3} e_0^2 \right) \right] \right. \\ &\quad \left. + (4s_0^2 - 1)e_0\xi \right\} \\ &\quad [\sin(v + 2\omega) + \sin(3v + 2\omega)] \\ &\quad + \frac{1}{6} j_0 \sigma_0 e_0^2 s_0^2 [\sin 2\omega + \sin(4v + 2\omega)] \end{aligned}$$

and

$$\begin{aligned} \frac{d\omega}{dv} &= \frac{1}{2} j_0 (5e_0^2 - 1) + j_0 \sigma_0 \\ &\quad \left[ -\frac{5}{4} - \frac{5}{24} e_0^2 + e_0^2 \left( \frac{23}{4} + \frac{9}{8} e_0^2 \right) \right] \\ &\quad + j_0 \xi \left[ \frac{15}{2} e_0^2 - 1 + e_0 \left( \frac{9}{2} e_0^2 - \frac{1}{3} \right) \right] \\ &\quad \cos v - \frac{3}{2} e_0 e_0^2 \cos(v + 2\omega) \\ &\quad - 3e_0^2 \cos(2v + 2\omega) - \frac{3}{2} e_0^2 e_0 \cos(3v + 2\omega) \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{12} j_0 \sigma_0 e_0^2 c_0^2 \cos 2\omega \\
& + \left[ \frac{1}{3} e_0 \sigma_0 + \sigma_0^2 \left( -\frac{1}{3} + e_0 - \frac{1}{9} e_0^2 \right) \right] \\
& \cos v + \frac{1}{12} e_0^2 \sigma_0^2 \cos 2v \\
& + j_0 c_0^2 \left\{ \left[ -1 - \sigma_0 \left( 2 + \frac{1}{2} e_0^2 \right) \right] \cos (2v + 2\omega) \right. \\
& + \left[ e_0 + \sigma_0 \left( -1 + \frac{4}{3} e_0 - \frac{1}{3} e_0^2 \right) \right] \\
& \left[ \cos v - \frac{1}{2} \cos (v + 2\omega) \right. \\
& \left. \left. - \frac{1}{2} \cos (3v + 2\omega) \right] + \frac{1}{6} \sigma_0 e_0^2 \cos 2v \right. \\
& \left. \left. - \frac{1}{12} \sigma_0 e_0^2 \cos (4v + 2\omega) \right\} \quad (26)
\end{aligned}$$

where

$$\left. \begin{aligned} j_0 &= \frac{J}{p_0^2} \\ \sigma_0 &= \frac{1}{2} j_0 (3c_0^2 - 1) \end{aligned} \right\} \quad (27)$$

and the initial conditions are

$$\left. \begin{aligned} \xi &= 0 \\ \omega &= \omega_0 \end{aligned} \right\} \text{ when } v=0 \quad (28)$$

Although these equations appear quite formidable, it is possible to simplify them without sacrificing any of the essential features of their solution. Thus, begin by obtaining the solution of the  $\xi$  equation, accurate to  $O(J)$ , with  $\omega$  being treated as a constant. This approximate solution is

$$\begin{aligned}
\xi \approx j_0 \kappa_0^2 & \left[ \left( 1 + \frac{4}{3} e_0 \right) \cos 2\omega - e_0 \cos (v + 2\omega) \right. \\
& \left. - \cos (2v + 2\omega) - \frac{1}{3} e_0 \cos (3v + 2\omega) \right]
\end{aligned}$$

If this is substituted back into the right members, but only in those terms in which  $\xi$  is multiplied by a trigonometric function containing the true anomaly,  $v$ , in its argument, the resulting equations take the form, to  $O(J^2)$ :

$$\left. \begin{aligned} \frac{dx}{dv} &= -b \sin 2\omega + \text{S.P.T.} \\ \frac{d\omega}{dv} &= x + a \cos 2\omega + \text{S.P.T.} \end{aligned} \right\} \quad (29)$$

where S.P.T. denotes short-period terms of the form  $\sin (mv + n\omega)$  or  $\cos (mv + n\omega)$ , with  $m \neq 0$ , and the quantities  $x$ ,  $a$ , and  $b$  are defined by:

$$\left. \begin{aligned} x &= x_0 + j_0 \left( \frac{15}{2} c_0^2 - 1 \right) \xi \\ x_0 &= \frac{1}{2} j_0 (5c_0^2 - 1) + j_0^2 \left[ \frac{5}{8} + \frac{5}{48} e_0^2 + c_0^2 \right. \\ & \quad \left. \left( -\frac{13}{4} + \frac{1}{8} e_0^2 \right) + c_0^4 \left( \frac{57}{8} + \frac{11}{16} e_0^2 \right) \right] \\ a &= j_0^2 e_0^2 \left( \frac{1}{6} - \frac{19}{8} c_0^2 + \frac{17}{8} c_0^4 \right) \\ b &= \frac{1}{12} j_0^3 e_0^2 (3c_0^2 - 1) (1 - c_0^2) \left( 1 - \frac{15}{2} c_0^2 \right) \end{aligned} \right\} \quad (30)$$

The coefficient,  $b$ , is  $O(J^3)$  since the definition of  $x$  has introduced one additional power of  $J$ .

The core of the problem consists in solving the equations obtained when the short-period terms are neglected; that is,

$$\left. \begin{aligned} \frac{dx}{dv} &= -b \sin 2\omega \\ \frac{d\omega}{dv} &= x + a \cos 2\omega \end{aligned} \right\} \quad (31)$$

with the initial conditions

$$\left. \begin{aligned} x &= x_0 \\ \omega &= \omega_0 \end{aligned} \right\} \text{ when } v=0 \quad (32)$$

#### THE PHASE-PLANE INTEGRAL

If the variable,  $x$ , is eliminated from equations (31) by differentiating the second and subtracting the first, the resulting equation can be put in the form

$$z \frac{dz}{d\omega} = -\frac{1}{2} \gamma^2 (1 + z) \sin 2\omega \quad (33)$$

where

$$\left. \begin{aligned} z &= \frac{2a}{b} \frac{d\omega}{dv} \\ \gamma^2 &= \frac{8a^2}{b} \end{aligned} \right\} \quad (34)$$

and the initial conditions are

$$\left. \begin{aligned} \omega &= \omega_0 \\ z &= z_0 = \frac{2a}{b} (x_0 + a \cos 2\omega_0) \end{aligned} \right\} \text{ at } v=0 \quad (35)$$

Separating the variables in equation (33) yields the phase plane integral

$$z - \ln(1+z) = \frac{1}{2} \gamma^2 (k^2 - \sin^2 \omega) \quad (36)$$

where the constant of integration,  $k^2$ , is given in terms of the initial values by

$$k^2 = \sin^2 \omega_0 + \frac{2}{\gamma^2} [z_0 - \ln(1+z_0)] \quad (37)$$

Elimination of the constant,  $k^2$ , gives the phase-plane integral the alternative form

$$z - z_0 - \ln \frac{1+z}{1+z_0} = \frac{1}{2} \gamma^2 (\sin^2 \omega_0 - \sin^2 \omega) \quad (38)$$

This form will be useful in those regions in which the constants  $\gamma$  and  $k$  turn out to be complex numbers.

The phase-plane integral, equation (36), is an implicit differential equation, since  $z$  contains the derivative  $d\omega/dv$ . The first problem is to obtain  $z$  explicitly as a function of  $\omega$ . To do this, introduce a new variable,  $w$ , defined by

$$w^2 = \gamma^2 (k^2 - \sin^2 \omega) \quad (39)$$

Then the differential equation (33) becomes

$$z \frac{dz}{dw} = w(1+z) \quad (40)$$

and the integral, equation (36), becomes

$$z - \ln(1+z) = \frac{1}{2} w^2 \quad (41)$$

Expanding the logarithm in a Maclaurin's series gives, near the "origin" ( $z=0$ ,  $w=0$ )

$$w^2 = z^2 - \frac{2}{3} z^3 + \frac{2}{4} z^4 - \frac{2}{5} z^5 \dots$$

so that  $w(z)$  and  $z(w)$  are both double-valued functions, with a branch point at the origin. It then follows that  $z$  is a double-valued function of

$\omega$ , and so is  $d\omega/dv$ . It is only this condition that is of any physical significance, since  $w$  occurs in the equations only in the squared form. Consequently, it is permissible to choose, arbitrarily, one of the branches of  $z(w)$ , as long as both branches of  $w(\omega)$  are retained. The branch of  $z(w)$  that will be chosen here is the one for which  $z/w \rightarrow 1$  as  $w \rightarrow 0$ .

Equation (40) can be solved formally to give  $z(w)$  as a Maclaurin's series:

$$z = \sum_{n=1}^{\infty} z_n w^n \quad (42)$$

where the coefficients,  $z_n$ , satisfy the recurrence relations

$$\left. \begin{aligned} z_1 &= 1 \\ z_2 &= \frac{1}{3} \\ (n+1)z_n &= z_{n-1} - \sum_{k=1}^{n-2} (k+1)z_{k+1}z_{n-k}, \quad n \geq 3 \end{aligned} \right\} \quad (43)$$

The first few coefficients are

$$z_3 = \frac{1}{36}, \quad z_4 = -\frac{1}{270}, \quad z_5 = \frac{1}{4320}, \quad z_6 = \frac{1}{17010} \dots$$

The other branch of  $z(w)$  is obtainable simply by changing the sign of each odd-numbered coefficient,  $z_n$ , and the only effect of such a choice would be to interchange the two branches of  $d\omega/dv$  considered as a function of  $\omega$ .

The radius of convergence of the series (42) can be obtained by finding the singularity of  $z(w)$  that is closest to the origin in the complex plane. From equation (40) it is evident that singularities occur ( $dz/dw = \infty$ ) when  $z=0$ ,  $w \neq 0$ . Since the logarithm is a multiple-valued function, equation (41) gives the desired radius of convergence,  $w_c$ , as

$$w_c = \sqrt[4]{2|\ln 1|} = \sqrt[4]{2|2\pi i|} = 2\sqrt{\pi} \quad (44)$$

Equation (41) gives the region of convergence as:

$$-0.9993 < z_0 < 8.5385 \quad (45)$$

Clearly, the quantities  $\gamma$  and  $w$  will be real only if

$$\left. \begin{aligned} b &> 0 \\ 1+z &> 0 \end{aligned} \right\} \quad (46)$$

simultaneously. It can be shown that these conditions are satisfied in the region

$$62^{\circ}.6 < I_0 < 68^{\circ}.6 \quad (47)$$

for eastbound orbits. For westbound orbits,  $I_0$  should be replaced by  $180^{\circ} - I_0$  throughout.

The upper limit is independent of the oblateness parameter,  $J$ , and of the elements of the orbit. The lower limit is relatively insensitive to these quantities; for example, it becomes  $62^{\circ}.7$  in the extreme case of  $J=0.1$ ,  $e_0=1$ ,  $p_0=2$ , the nominal values for a grazing, parabolic orbit about a highly oblate planet like Saturn.

Since these boundaries correspond to  $-1 < z_0 < \infty$ , it is clear that the region of convergence lies inside them (see eq. (45)). Neglecting higher order terms in  $J$  gives the region of convergence as

$$62^{\circ}.61 < I_0 < 67^{\circ}.41 \quad (48)$$

and both these bounds are relatively insensitive to the oblateness and orbital parameters. Thus, for a grazing, parabolic orbit about Saturn the region of convergence is

$$62^{\circ}.73 < I_0 < 67^{\circ}.48 \quad (49)$$

This region of convergence will be called the "elliptic region," since it will appear that the solution in this region can be expressed in terms of elliptic functions. Similarly, the remaining region will be called the "trigonometric region."

The phase-plane integral, equation (36), is shown schematically in figure 2 for the elliptic region. No scale is shown on the vertical axis, since this depends on the oblateness parameter,  $J$ , and the orbital parameters  $e_0$  and  $p_0$  in a non-linear fashion. Each curve represents a fixed value of  $k$ .

Actually, the initial values of the apsidal position and velocity cannot be assigned arbitrarily for any phase-plane curve of constant  $k$ , since they are related to the other parameters  $I_0$ ,  $e_0$ , and  $p_0$  through equations (30) through (32). Thus, if  $\omega_0$ ,  $I_0$ ,  $e_0$ , and  $p_0$  are specified a priori, then  $k$  and  $(d\omega/dr)_0$  are determined, and the orbit is represented by the corresponding curve in the phase plane. It will be seen later that, as the curve is traversed, the parameters  $I$  and  $p$  change. Thus, any attempt to regard an arbitrary point as a new "initial value" would require

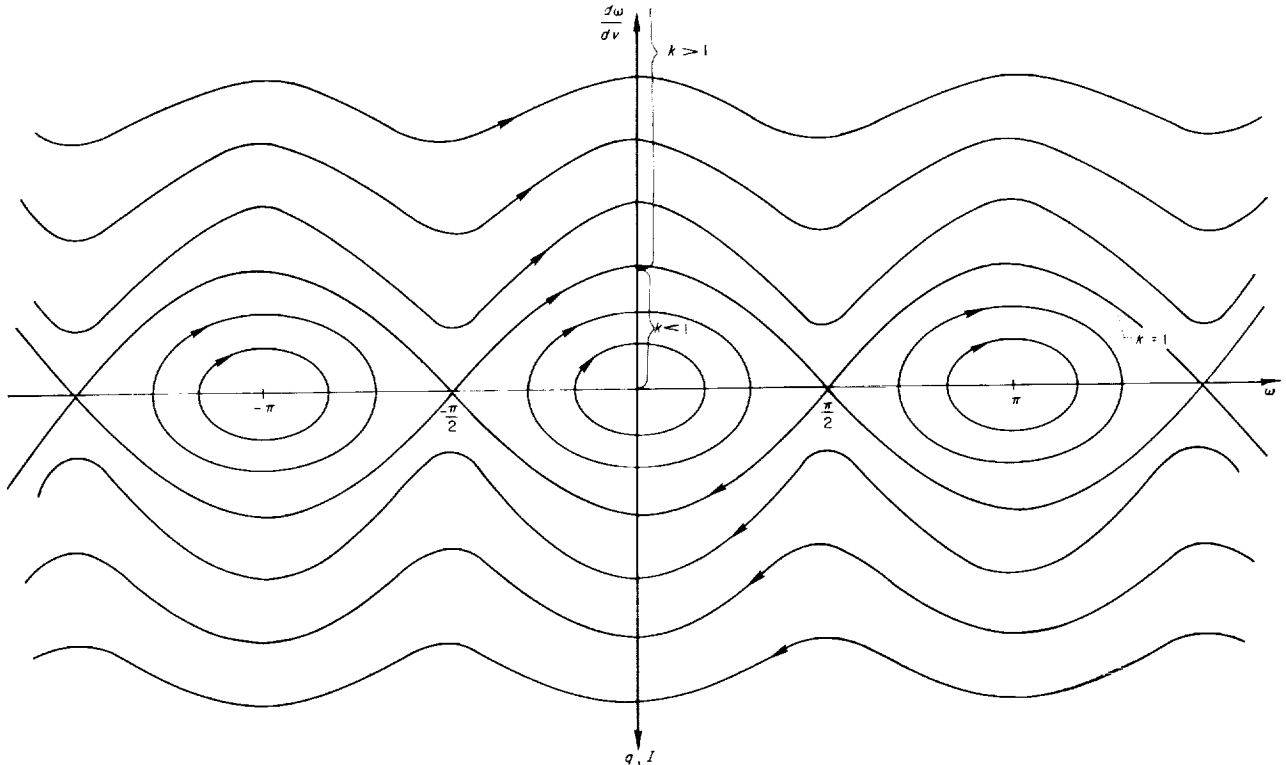


FIGURE 2.—The Phase-plane integral.

recomputing the parameters  $x_0$ ,  $a$ , and  $b$ , and hence would lead to a new value of  $k$ , and thus to a new curve in the phase plane.

However, the two curves would differ at most by terms of  $O(\sqrt{J})$ ; figure 2 can still be regarded as qualitatively correct, and certain features of the orbits can be deduced from it.

Thus, when  $k < 1$ , the argument of pericenter,  $\omega$ , is bounded and periodic, being restricted to the region given by  $|\sin \omega| \leq k$ . Pericenter oscillates about the node and never reaches the points of maximum declination. This is known as the libration region.

When  $k > 1$ , the apsidal velocity,  $d\omega/dv$ , never changes sign. The apsidal motion is secular, with periodic oscillations superimposed. It will be seen later that this is true throughout the trigonometric region also. This is known as the secular region.

When  $k = 1$ , the apsidal motion is aperiodic. The argument of pericenter approaches monotonically the point of maximum declination; it will be shown later that this approach takes infinite time. This limiting case is known as the separatrix, and the limiting points as saddle points.

The curve  $k = 0$  reduces to a single point (see fig. 2), at which  $\omega = 0$  or  $\pi$  (ascending or descending node), and  $d\omega/dv = 0$ ; that is, there is no apsidal motion. This occurs when

$$x_0 + a = 0$$

and the angle of inclination at which this occurs is called the critical angle, denoted by  $I_c$ . Neglecting higher powers of  $J$  gives the approximation

$$\left. \begin{aligned} \cos^2 I_c &\cong \frac{1}{5} \\ I_c &\cong 63^\circ.435 \end{aligned} \right\} \quad (50)$$

and again this quantity is relatively insensitive to the orbital and oblateness parameters. Thus, for grazing, parabolic orbits about Earth and Saturn, the values are  $63^\circ.437$  and  $63^\circ.576$ , respectively.

The series expansion (eq. (42)) is one representation of  $d\omega/dv$  as a function of  $\omega$ , and its region of convergence has been determined, namely the elliptic region given by equation (45) or (48). It is now desired to obtain a solution outside the elliptic region. To do this it is convenient to

make the following transformations of variables:

$$\left. \begin{aligned} \zeta &= \frac{z - z_0}{1 + z_0} \\ z &= z_0 + \zeta(1 + z_0) \\ \lambda &= \frac{w^2 - w_0^2}{2z_0} = \frac{a}{\omega_0'} (\cos 2\omega - \cos 2\omega_0) \\ \omega_0' &= \left( \frac{d\omega}{dv} \right)_{v=0} \end{aligned} \right\} \quad (51)$$

Then the differential equation (33) and the phase-plane integral (38) become

$$\left[ 1 + \zeta \left( 1 + \frac{1}{z_0} \right) \right] \frac{d\zeta}{d\lambda} = 1 + \zeta \quad (52)$$

and

$$\lambda - \zeta \left( 1 + \frac{1}{z_0} \right) - \frac{1}{z_0} \ln(1 + \zeta) \quad (53)$$

respectively.

A formal solution of equation (52) is given by the series

$$\zeta = \sum_{n=1}^{\infty} \zeta_n \lambda^n \quad (54)$$

where the coefficients,  $\zeta_n$ , satisfy the recurrence relations

$$\left. \begin{aligned} \zeta_1 &= 1 \\ (n+1)\zeta_{n+1} &= \zeta_n - \left( 1 + \frac{1}{z_0} \right) \sum_{k=0}^{n-1} (k+1)\zeta_{k+1}\zeta_{n-k}, \\ &\quad n \geq 1 \end{aligned} \right\} \quad (55)$$

The first few coefficients are

$$\begin{aligned} \zeta_2 &= -\frac{1}{2z_0} \\ \zeta_3 &= \frac{1}{3z_0} + \frac{1}{2z_0^2} \\ \zeta_4 &= -\frac{1}{4z_0} - \frac{5}{6z_0^2} - \frac{5}{8z_0^3} \end{aligned}$$

Again the radius of convergence is obtained by finding the singularity in the complex plane ( $d\zeta/d\lambda = \infty$ ) nearest the origin. Equation (52) gives

$$\zeta = \zeta_c \equiv \frac{-z_0}{1 + z_0}$$

and then equation (53) gives

$$\lambda = \lambda_c \equiv -1 + \frac{1}{z_0} \ln(1 + z_0)$$

and there are two cases, depending on the sign of  $(1+z_0)$ .

Thus, the radius of convergence,  $\lambda_c$ , is

$$\left. \begin{aligned} \lambda_c &= \left| -1 + \frac{1}{z_0} \ln(1+z_0) \right| & \text{if } 1+z_0 > 0 \\ \lambda_c &= \sqrt{\left[ -1 + \frac{1}{z_0} \ln(-1-z_0) \right]^2 + \left( \frac{\pi}{z_0} \right)^2} & \text{if } 1+z_0 < 0 \end{aligned} \right\} \quad (56)$$

In particular,

$$\left. \begin{aligned} \lambda_c &= 0 & \text{if } z_0 = 0 \\ \lambda_c &= \infty & \text{if } z_0 = -1 \end{aligned} \right\} \quad (57)$$

Note that in the elliptic region, where  $1+z_0 > 0$ , the radius of convergence,  $\lambda_c$ , can be put in the form

$$\lambda_c = \left| \frac{w_0^2}{2z_0} \right| \quad (58)$$

Next it will be shown that the two regions of convergence, elliptic and trigonometric, do overlap and do cover all possible cases. This will be done by obtaining simplified expressions exhibiting the dependence of the various parameters  $\gamma$ ,  $k$ ,  $\lambda_c$ , etc., on the oblateness parameter,  $J$ , and the orbital parameters,  $\omega_0$ ,  $e_0$ , and  $p_0$ , and by giving typical numerical examples based on exact calculations.

Throughout, the oblateness parameter,  $J$ , will be subjected to the restriction

$$J < 0.25$$

When reference is made to Earth or Saturn, the calculations are based on the nominal values

$$J = \begin{cases} 0.0016 & \text{for Earth} \\ 0.1 & \text{for Saturn} \end{cases}$$

To obtain bounds for certain quantities, it is convenient to extend the class of satellite orbits to include escape orbits. In particular, the parameter  $2e_0/p_0$  will occur frequently. Since

$$p_0 = q_0(1+e_0)$$

where  $e_0$  is the eccentricity,  $p_0$  the semilatus rectum, and  $q_0$  the pericenter distance, it is clear

that, for the limiting case of grazing, parabolic orbits,

$$e_0 = 1$$

$$q_0 = 1$$

$$p_0 = 2$$

$$2e_0/p_0 = 1$$

and, for satellite orbits,

$$e_0 < 1$$

$$q_0 > 1$$

$$\frac{2e_0}{p_0} < \frac{2e_0}{1+e_0} < 1$$

The analysis will be carried out only for east-bound orbits,  $0^\circ \leq I_0 \leq 90^\circ$ . All the results are valid for westbound orbits when  $I_0$  is replaced by  $180^\circ - I_0$  throughout.

Consider first the parameter,  $\gamma$ , which plays the role of a scale factor:

$$|\gamma| = \left| 2a \sqrt{\frac{2}{b}} \right|$$

Taking  $\cos^2 I_0 = 1/5$  gives the approximation

$$|\gamma| = 2.75 \left( \frac{2e_0}{p_0} \right) \sqrt{J} \quad (59)$$

Actual values are shown in figure 3 for grazing, parabolic orbits. For comparison, equation (59) gives

$$|\gamma| = \begin{cases} 0.11 & \text{for Earth} \\ 0.87 & \text{for Saturn} \end{cases}$$

Thus, in the elliptic region,  $|\gamma|$  is essentially independent of the inclination angle, so that it is, in fact, a scale factor.

Next, consider the parameter,  $k$ , which will subsequently be identified as the modulus of certain elliptic functions. Setting  $\omega = \omega_0 = 0$  or  $\pi$  in equation (39) (fig. 2 shows this is permissible without any loss of generality) gives

$$k = \left| \frac{w_0}{\gamma} \right|$$

Since  $w_0$  is essentially a function of the inclination

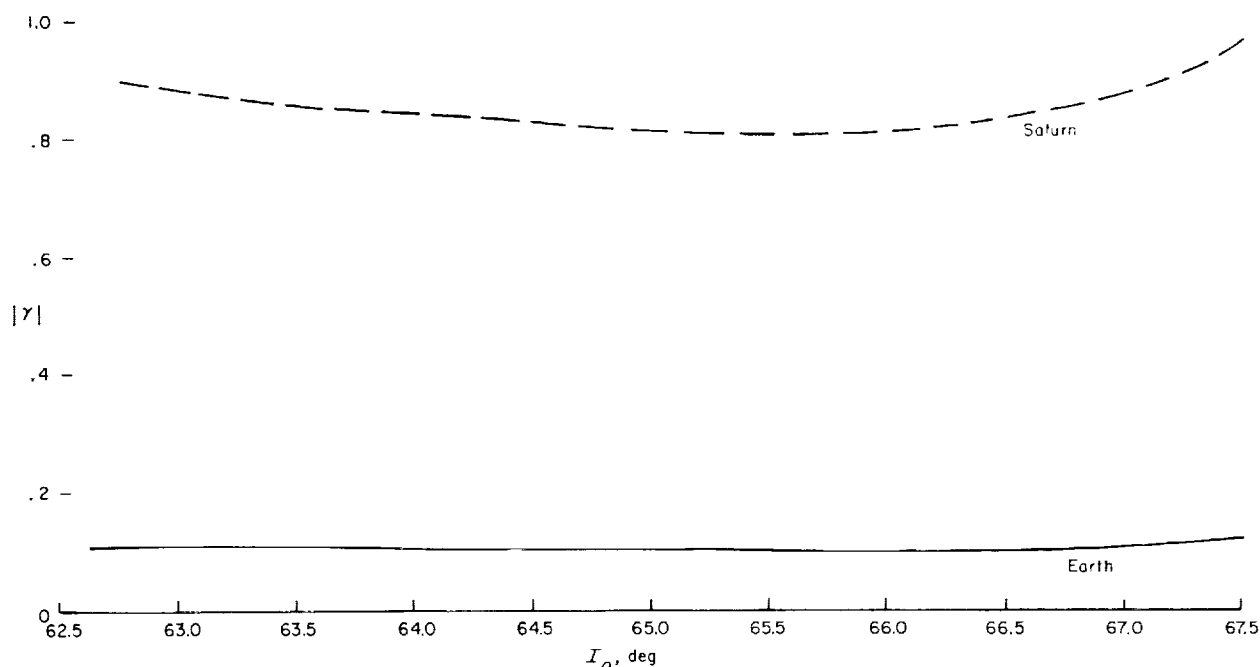


FIGURE 3.—Oblateness scale factor for grazing parabolic orbits.

angle only, equation (59) gives the functional form

$$k \cong \frac{p_0}{e_0 \sqrt{J}} f(I_0) \quad (60)$$

Numerical values are presented in figure 4, where, for convenience, the reciprocal,  $1/k$ , is plotted in the secular region.

The series expansion (42) was shown to have the radius of convergence  $w_c = 2\sqrt{\pi}$ . Hence, by the Cauchy-Hadamard theorem (ref. 9, pp. 154-5),

$$\frac{1}{w_c} = \lim_{n \rightarrow \infty} \sqrt[n]{|z_n|}$$

Thus (ref. 9, p. 91), for any  $\epsilon$ ,

$$\sqrt[n]{|z_n|} < \frac{1}{w_c} + \epsilon$$

for all  $n > N(\epsilon)$ . Hence

$$|z_n w_0^n| < \left( \left| \frac{w_0}{w_c} \right| + \epsilon |w_0| \right)^n$$

and the speed of convergence is determined essentially by the value of the convergence parameter  $|w_0/w_c|$ . This parameter depends essentially on the inclination angle. Numerical values

are presented in figure 5, which shows clearly the insensitivity to the oblateness parameter,  $J$ .

Similarly, for the trigonometric region the natural convergence parameter is  $\lambda/\lambda_c$ . By equation (56), the radius of convergence,  $\lambda_c$ , is essentially a function of inclination angle only, while, by equation (51)

$$|\lambda| \leq \frac{2a}{\omega_0} \cong \frac{2a}{x_0}$$

Equations (31) for  $x_0$  and  $a$  then give the proportionality, for fixed but arbitrary  $I_0$ ,

$$\left| \frac{\lambda}{\lambda_c} \right|_{\max} \sim J \left( \frac{e_0}{p_0} \right)^2 \quad (61)$$

for the dependence on  $J$ ,  $e_0$ , and  $p_0$ . Numerical values for the region outside the elliptic region are presented in figure 6, with scales at left and right for Earth and Saturn, respectively. Thus, outside the elliptic region,

$$\left| \frac{\lambda}{\lambda_c} \right| \leq \begin{cases} 0.0024 & \text{for Earth} \\ 0.15 & \text{for Saturn} \end{cases} \quad (62)$$

and the speed of convergence of the series (54) is quite satisfactory.

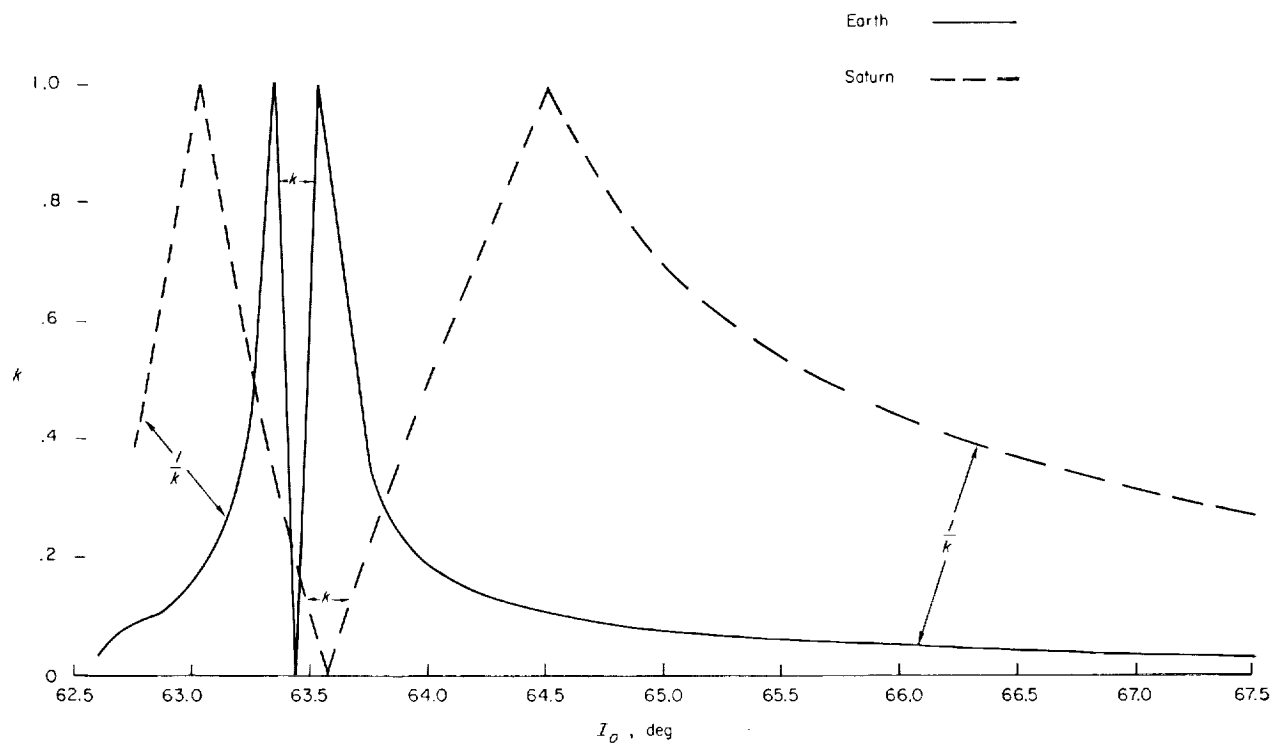


FIGURE 4—Nodus of elliptic functions for grazing parabolic orbit.

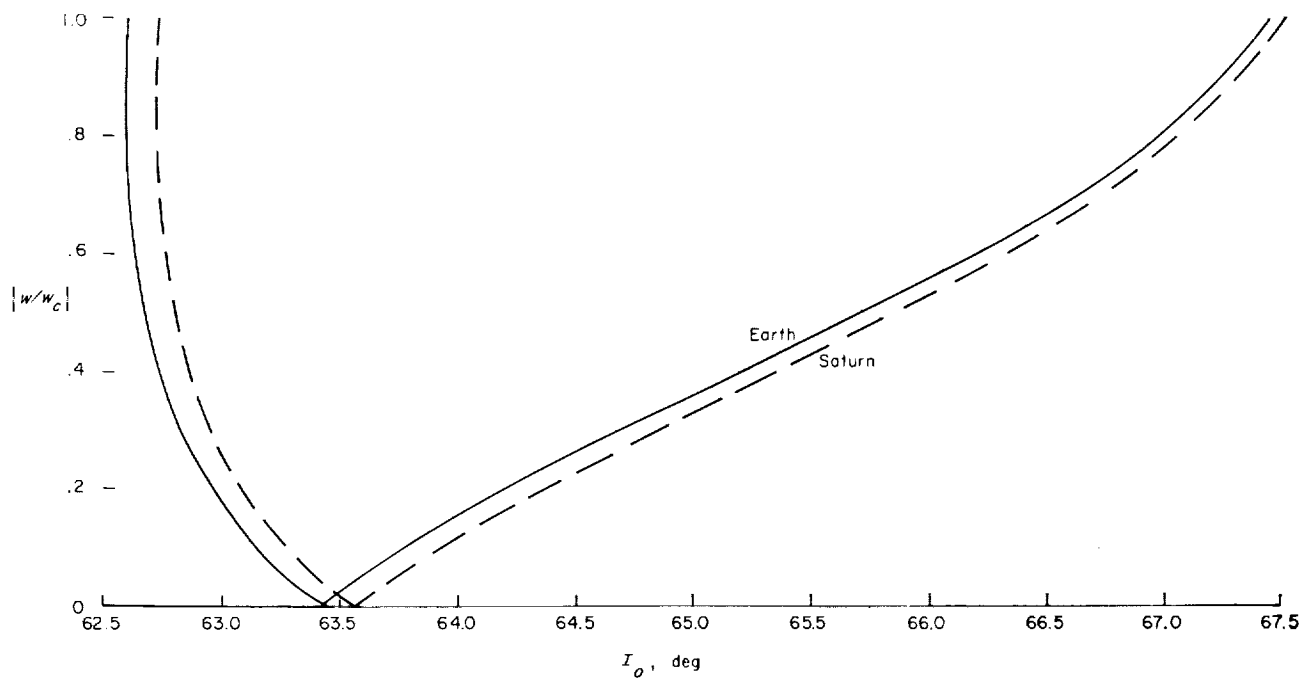


FIGURE 5.—Convergence parameter in the elliptic region.

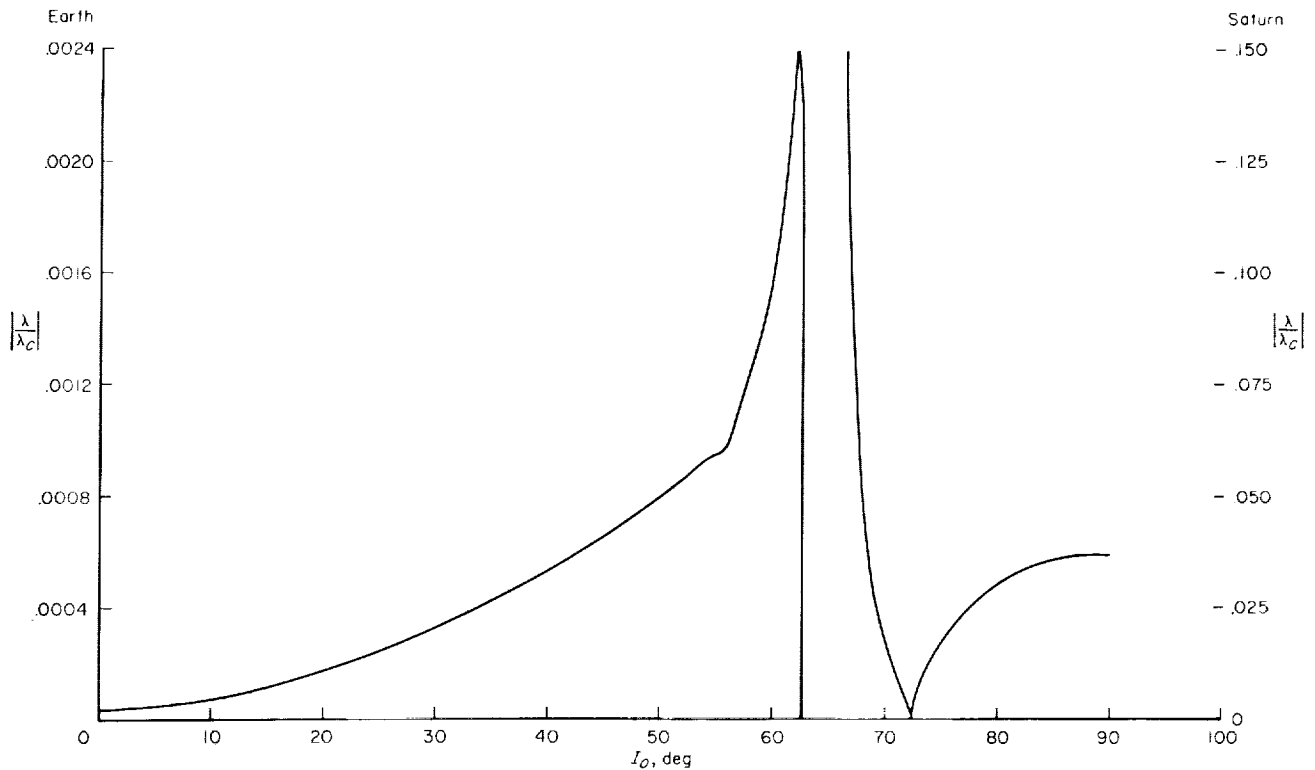


FIGURE 6.—Trigonometric convergence factor in trigonometric region for grazing parabolic orbits.

It remains to be shown that the elliptic and trigonometric regions do indeed overlap. Consider, first, the trigonometric convergence factor at the boundary of the elliptic region. By equations (51) and (58)

$$\left| \frac{\lambda}{\lambda_c} \right| = \left| \frac{w^2 - w_0^2}{w_0^2} \right| \quad (63)$$

At the boundary,  $w_0^2 = 4\pi$  and equation (51) then gives

$$\left| \frac{\lambda}{\lambda_c} \right| \leq \frac{\gamma^2}{4\pi} = 0.6J \left( \frac{2e_0}{p_0} \right)^2 < 1 \quad (64)$$

the second inequality following from equation (59). Thus the trigonometric series does converge at the boundary of the elliptic region.

Conversely, the boundary of the trigonometric region can be obtained from equation (63) by inserting  $w$  and  $w_0$  from equation (39):

$$\left| \frac{\lambda}{\lambda_c} \right| = \left| \frac{\gamma^2 (\sin^2 \omega_0 - \sin^2 \omega)}{\gamma^2 (k^2 - \sin^2 \omega_0)} \right| \quad (65)$$

Taking the worst case,  $\omega_0 = \pi/2$ ,  $\omega = 0$  gives

$$\left| \frac{\lambda}{\lambda_c} \right| \leq \left| \frac{1}{k^2 - 1} \right| \quad (66)$$

Thus, the convergence boundary for the trigonometric series is  $k^2 = 2$ , and here the elliptic convergence factor is, by equations (39) and (59),

$$\left| \frac{w}{w_c} \right| \leq \left| \frac{\gamma}{\sqrt{2\pi}} \right| \leq 1.1\sqrt{J} \left( \frac{2e_0}{p_0} \right) < 1$$

and the overlapping of the convergence regions has been established.

It is of interest to determine the speeds of convergence in that region common to both the elliptic and trigonometric regions, and to determine the point at which the speeds are equal. Equations (39) and (66) give

$$\left| \frac{\lambda}{\lambda_c} \right| \leq \frac{1}{k^2 - 1} = \left| \frac{k\gamma}{2\sqrt{\pi}} \right| \geq \left| \frac{w}{w_c} \right| \quad (67)$$

Equation (59) then gives

$$k^3 - k = \frac{2\sqrt{\pi}}{|\gamma|} = \frac{1.3}{\sqrt{J}} \frac{p_0}{2e_0} \quad (68)$$

This gives, in the worst case of grazing, parabolic orbits, for Earth, Saturn, and the extreme case  $J=0.25$ ,

$$\left. \begin{array}{l} k=3.32, 1.8, 1.62 \\ \left| \frac{\lambda}{\lambda_c} \right|, \left| \frac{w}{w_c} \right| \leq 0.10, 0.45, 0.62 \end{array} \right\} \quad (69)$$

respectively, and the dependence on  $J$ ,  $e_0$ ,  $p_0$  is given approximately by the equation

$$\left| \frac{\lambda}{\lambda_c} \right| \text{ or } \left| \frac{w}{w_c} \right| = \sqrt[3]{\left( \frac{2e_0}{p_0} \right)^2 J} \quad (70)$$

(This approximation gives 0.12, 0.46, 0.63 in eq. (69)).

Thus, in the case of the Earth, one can always choose a series that converges more rapidly than

$$\sum_{n=1}^{\infty} \left[ \frac{1}{10} \left( \frac{2e_0}{p_0} \right)^{2/3} \right]^n$$

and this worst case occurs only in the common region. In the libration and purely trigonometric regions the corresponding, dominating series are

$$\sum_{n=1}^{\infty} \left( 0.031 \frac{2e_0}{p_0} \right)^n$$

and

$$\sum_{n=1}^{\infty} \left[ 0.0024 \left( \frac{2e_0}{p_0} \right)^2 \right]^n$$

respectively; hence, ruthless truncation of the series is justified for Earth satellite orbits, especially for those of moderate eccentricity!

A single convergence parameter,  $\nu$ , can be defined as follows:

$$\nu = \text{minimum} \left( \left| \frac{w}{w_c} \right|, \left| \frac{\lambda}{\lambda_c} \right| \right)$$

in the overlapping region, while  $\nu$  is defined to be  $|w/w_c|$  or  $|\lambda/\lambda_c|$  in the purely elliptic or purely trigonometric region, respectively. Then the geometric series

$$\sum_{n=1}^{\infty} \nu^n$$

is a dominant for all the series. This "optimum convergence parameter,"  $\nu$ , is shown in figure 7, for the overlapping region.

It should be emphasized that the quantities  $\sqrt{J}$  and  $\sqrt[3]{J}$  play distinct significant roles. As will be seen (e.g., eq. (89)), the width of the libration region (range of inclination angle) is proportional to  $\sqrt{J}$ , and this region disappears as  $J$  approaches zero. Also, the secular velocity of pericenter vanishes with  $J$ . However, the speed of convergence of the series in the overlapping region is proportional to  $\sqrt[3]{J}$ , by equation (70), and this

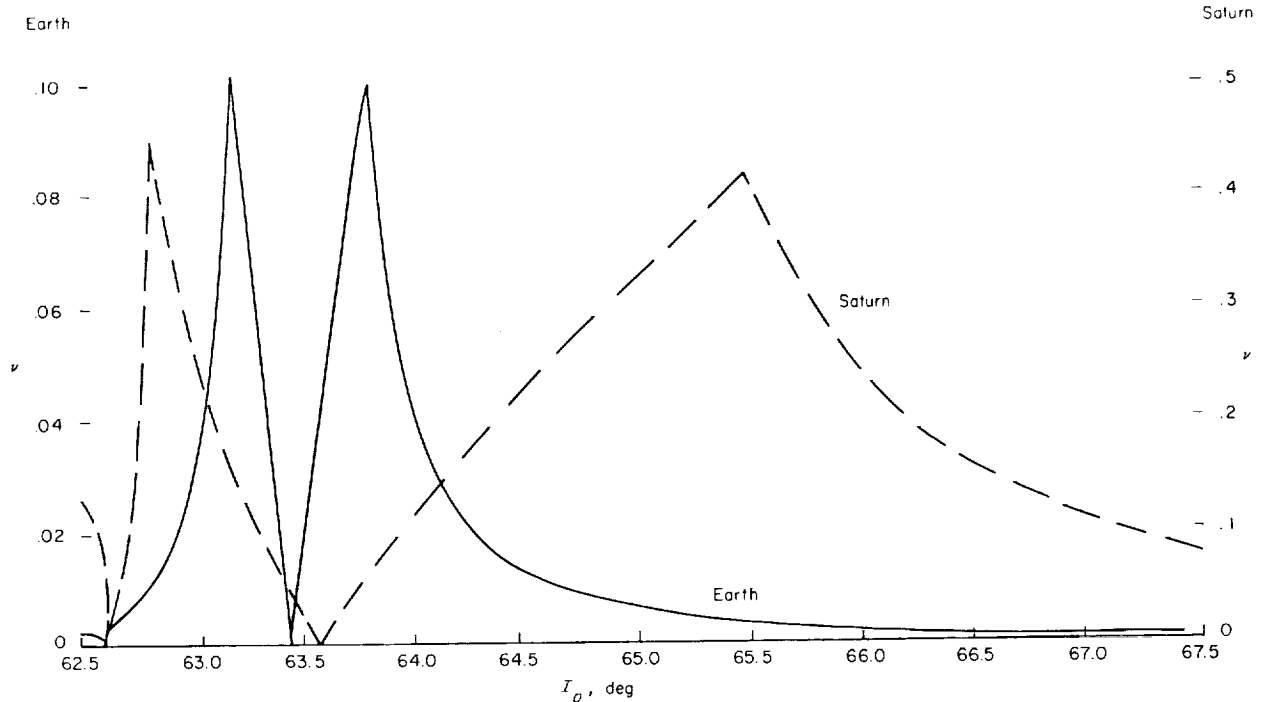


FIGURE 7.--Optimum convergence factor.

determines the number of terms of the series required for a specified accuracy.

The series representations of the phase-plane integral, equations (42) and (54), are of the form

$$\frac{d\omega}{dv} = \text{series in } \omega$$

It would be more convenient to express the reciprocal,  $dv/d\omega$ , in this form. This can easily be done. For the elliptic region, equations (34) and (40) yield

$$\frac{dv}{d\omega} = \frac{2a}{b} \left( \frac{1}{w} \frac{dz}{dw} - 1 \right)$$

The transformation equations (51) then yield, for the trigonometric region,

$$\frac{dv}{d\omega} = \frac{1}{\omega_0'} \left[ (1+z_0) \frac{d\xi}{d\lambda} - z_0 \right]$$

Inserting the series expansions for  $z$  and  $\xi$  gives the desired equations

$$\left. \begin{aligned} \frac{dv}{d\omega} &= \frac{2a}{b} \left[ \frac{1}{w} \sum_{n=0}^{\infty} (n+1) z_{n+1} w^n - 1 \right] \\ w^2 &= \gamma^2 (k^2 - \sin^2 \omega) \end{aligned} \right\} \quad (71)$$

for the elliptic region, and

$$\left. \begin{aligned} \frac{dv}{d\omega} &= \frac{1}{\omega_0'} \left[ (1+z_0) \sum_{n=0}^{\infty} (n+1) \xi_{n+1} \lambda^n - z_0 \right] \\ \lambda &= \frac{a}{\omega_0'} (\cos 2\omega - \cos 2\omega_0) \end{aligned} \right\} \quad (72)$$

for the trigonometric region.

In the following sections these equations will be solved, yielding the implicit solution

$$v = v(\omega)$$

This will then be inverted to obtain

$$\omega = \omega(v)$$

Equation (31) then yields

$$x = \frac{d\omega}{dt} - a \cos 2\omega \quad (73)$$

without any additional integration. It is then simply an exercise in algebra to obtain the angular momentum, semilatus rectum, and inclination angle as functions of the true anomaly,  $v$ .

## SOLUTION OF THE APSIDAL EQUATIONS IN THE ELLIPTIC REGION

In order to simplify the analysis, the initial position of pericenter will be taken at the ascending node,  $\omega_0 = 0$ . Figure 2 shows that this entails no loss of generality, except in the portion of the libration region centered at the descending node,  $\omega_0 = \pi$ . The results to be obtained can be applied to this case if  $\omega$  and  $\omega_0$  are simply replaced by  $\omega - \pi$  and  $\omega_0 - \pi$ , respectively.

To insure that both branches of the phase-plane integral are retained, it is sufficient to define the sign of the parameter,  $\gamma$ , of equation (34), by

$$\gamma = 2a \sqrt{\frac{2}{b}} \text{sign } \omega_0' \quad (74)$$

that is, the branches are identified simply by the sign of  $\omega_0'$  (see fig. 2).

It is now convenient to introduce a new independent variable,  $\psi$ , and a new dependent variable,  $\vartheta$ , defined by

$$\left. \begin{aligned} \psi &= r \sqrt{2b} \text{sign } \omega_0' \\ \sin \omega &= k \text{sn } \vartheta \end{aligned} \right\} \quad (75)$$

where  $\text{sn } \vartheta = \text{sn}(\vartheta, k)$  is the Jacobian elliptic function of modulus,  $k$  (which will not be written explicitly). Standard terminology for elliptic functions and integrals is used throughout (see, e.g., ref. 10). Equations (71), (74), and (75) now yield

$$\left. \begin{aligned} \cos \omega &= dn \vartheta \\ \frac{d\omega}{d\vartheta} &= k \text{cn } \vartheta \\ w &= k\gamma \text{cn } \vartheta \end{aligned} \right\} \quad (76)$$

and the differential equation

$$\frac{d\psi}{d\vartheta} = 1 - \frac{1}{3} k\gamma \text{cn } \vartheta + \sum_{n=2}^{\infty} (n+1) z_{n+1} (k\gamma \text{cn } \vartheta)^n \quad (77)$$

Integrating term-by-term gives

$$\psi = \vartheta - \frac{1}{3} \gamma I_1(\vartheta) + \sum_{n=2}^{\infty} (n+1) z_{n+1} \gamma^n I_n(\vartheta) \quad (78)$$

where

$$I_n(\vartheta) = \int_0^{\vartheta} (k \text{cn } \vartheta_1)^n d\vartheta_1, \quad n \geq 1 \quad (79)$$

These can be evaluated by means of the recurrence relations of reference 10, pages 192-193; the

first few are

$$\left. \begin{aligned} I_1(\vartheta) &= \arcsin(k \operatorname{sn} \vartheta) \\ I_2(\vartheta) &= Z(\vartheta) + \left( \frac{E}{K} + k^2 - 1 \right) \vartheta \\ I_3(\vartheta) &= \left( k^2 - \frac{1}{2} \right) \arcsin(k \operatorname{sn} \vartheta) \\ &\quad + \frac{1}{2} k \operatorname{sn} \vartheta \operatorname{dn} \vartheta \end{aligned} \right\} \quad (80)$$

where  $Z(\vartheta)$  is the Jacobian zeta function, and  $E, K$  are the complete elliptic integrals of the first and second kind.

Equation (78) may be regarded as defining  $\vartheta$  as a function of two independent variables  $\psi$  and  $\gamma$ . Differentiating with respect to  $\gamma$  gives

$$\frac{\partial \vartheta(\psi, \gamma)}{\partial \gamma} = \frac{z}{w} \left[ \frac{1}{3} I_1(\vartheta) - \sum_{n=2}^{\infty} n(n+1) z_{n+1} \gamma^{n-1} I_n(\vartheta) \right]$$

The series converges in the elliptic region, and  $z/w$  is bounded. Hence,  $\vartheta$  is an analytic function of  $\gamma$  throughout the elliptic region and can be expanded in a convergent Maclaurin series:

$$\vartheta = \psi + \sum_{n=1}^{\infty} J_n(\psi) \gamma^n \quad (81)$$

where

$$J_n(\psi) = \frac{1}{n!} \left[ \frac{\partial^n \vartheta(\psi, \gamma)}{\partial \gamma^n} \right]_{\gamma=0} \quad (82)$$

The first few are

$$\left. \begin{aligned} J_1(\psi) &= \frac{1}{3} I_1(\psi) \\ J_2(\psi) &= \frac{1}{9} k(\operatorname{cn} \psi) I_1(\psi) - \frac{1}{12} I_2(\psi) \\ J_3(\psi) &= -\frac{1}{54} k \operatorname{sn} \psi (\operatorname{dn} \psi) I_1^2(\psi) \\ &\quad + \frac{1}{108} k^2 (\operatorname{cn}^2 \psi) I_1(\psi) \\ &\quad - \frac{1}{36} k(\operatorname{cn} \psi) I_2(\psi) + \frac{2}{135} I_3(\psi) \end{aligned} \right\} \quad (83)$$

Thus the argument of pericenter,  $\omega$ , is expressed as an explicit function of  $v$ , the true anomaly, by equations (74), (75), and (81). The remaining variables to be determined are  $q$ , the pericenter distance,  $p$ , the semilatus rectum, and  $I$ , the inclination angle, all of which are obtainable, ultimately, from equation (73):

$$x = \frac{d\omega}{dv} - a \cos 2\omega \quad (73)$$

Combining this with equations (34), (75), and (76) gives

$$x - x_0 = \frac{b}{2a} \left[ z - z_0 - \frac{1}{2} (w^2 - w_0^2) \right] \quad (84)$$

and it may be recalled that

$$\begin{aligned} w &= k\gamma \operatorname{cn} \vartheta \\ w_0 &= k\gamma \end{aligned} \quad (76)$$

and  $z$  is a power series in  $w$  (eq. (42)). Equations (30) then give

$$\xi = \beta \left[ z - z_0 - \frac{1}{2} (w^2 - w_0^2) \right] \quad (85)$$

where

$$\beta = \frac{b}{2a j_0 \left( \frac{15}{2} c_0^2 - 1 \right)} = \frac{(1 - c_0^2)(1 - 3c_0^2)}{4 - 57c_0^2 + 51c_0^4} \quad (86)$$

Note that  $\beta$  depends only on the inclination angle,  $I_0$ , and is independent of the planet ( $J$ ) and of all other orbital parameters. Its graph is shown in figure 8, where it is called the inclination scale factor. The remaining variables are expressible in terms of  $\xi$  by equations (12) and (15):

$$\left. \begin{aligned} \text{Pericenter distance: } q &= \frac{p}{1 + e_0} \\ \text{Semilatus rectum: } p &= \frac{p_0}{1 + \xi} \\ \text{Inclination angle: } \cos^2 I &= \cos^2 I_0 (1 + \xi) \end{aligned} \right\} \quad (87)$$

The series presented in this and the previous section constitute the complete analytic solution of the problem in the elliptic region. The geometrical interpretation will now be given for the three cases

Libration region:  $0 \leq k < 1$

Separatrix:  $k = 1$

Secular region:  $k > 1$

In the interest of clarity and simplicity only the dominant terms of the series will be retained; as was remarked earlier, this is certainly justified in the most important case of earth satellites.

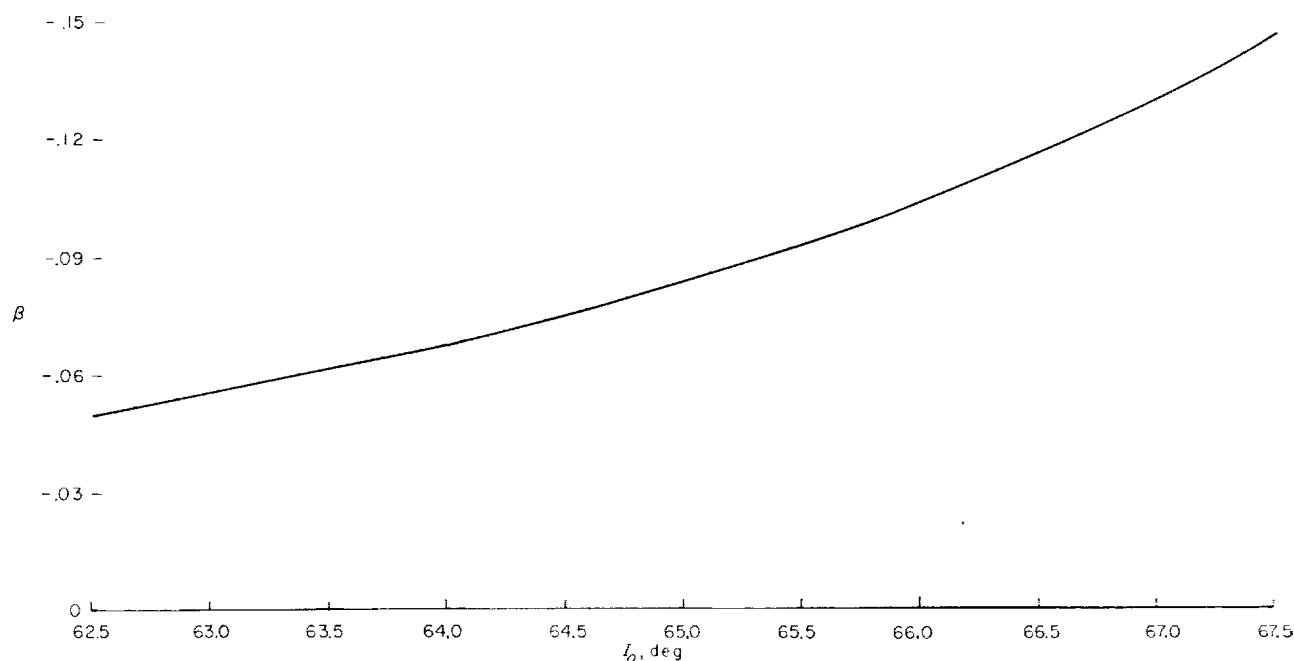


FIGURE 8.—Inclination scale factor.

THE LIBRATION REGION,  $0 \leq k < 1$ 

Figure 4 shows that  $I$  is essentially equal to the critical value throughout the libration region. Hence, the parameters  $a$ ,  $b$ ,  $\beta$ ,  $\gamma$  can be approximated by their values at  $I_0 = \arctan 2$ , and if  $\omega'_0 \geq 0$  (fig. 2 shows that this can be done without loss of generality), then truncation of the series of the preceding section gives

$$\left. \begin{aligned} \psi &= v\sqrt{2b} \\ \vartheta &= \psi + \frac{1}{3} \gamma \text{ arcs in } (k \sin \psi) \\ \sin \omega &= k \sin \vartheta \\ \frac{d\omega}{dv} &= k\sqrt{2b} \operatorname{cn} \vartheta \\ I &= I_c - k\Delta I \operatorname{cn} \vartheta \\ \frac{dI}{dv} &= \frac{1}{2} \Delta I \sqrt{2b} \sin 2\omega \\ q &= q_0 + k\Delta q (1 - \operatorname{cn} \vartheta) \\ \frac{dq}{dv} &= \frac{1}{2} \Delta q \sqrt{2b} \sin 2\omega \end{aligned} \right\} \quad (88)$$

with the following notation and approximations:

$$\left. \begin{aligned} I_c &= \arctan 2 + \frac{J}{q_0^2(1+e_0)^2} \left( \frac{13}{100} - \frac{e_0^2}{30} \right) \\ \gamma &= \frac{-67e_0\sqrt{6J}}{30q_0(1+e_0)} \\ \sqrt{2b} &= \frac{e_0J\sqrt{6J}}{15q_0^3(1+e_0)^3} \\ \Delta I &= \frac{-\gamma}{67} \\ \Delta q &= \frac{-4\gamma q_0}{67} \end{aligned} \right\} \quad (89)$$

The dependence of the critical angle,  $I_c$ , on the oblateness and orbital parameters is shown explicitly. For satellite orbits,  $0 \leq e_0 < 1$ , and, for grazing orbits ( $q_0 = 1$ ):

$$\left. \begin{aligned} 63^\circ.437 &\leq I_c \leq 63^\circ.447 \text{ for Earth} \\ 63^\circ.573 &\leq I_c \leq 64^\circ.180 \text{ for Saturn} \end{aligned} \right\} \quad (90)$$

The elliptic functions  $\operatorname{sn}$  and  $\operatorname{cn}$  are periodic; they are qualitatively similar to the trigonometric

functions sine and cosine, having unit amplitude (deviation from mean value). However, the period is  $4K(k)$ ; when  $k=0$ ,  $K=\pi/2$ , and as  $k$  increases so does  $K$ ; as  $k \rightarrow 1$ ,  $K \rightarrow \infty$  logarithmically (see ref. 10).

Thus  $\omega$ ,  $I$ , and  $q$  are all periodic functions of the true anomaly,  $v$ , with the same period  $P$ :

$$P = \frac{39Kq_0^3(1+e_0)^3}{\pi e_0 J \sqrt{6J}} \text{ orbital revolutions} \quad (91)$$

The minimum value occurs when  $k=0$ ,  $q_0=1$ ,  $e_0=1/2$ :

$$P \geq \frac{100}{J \sqrt{6J}} = \begin{cases} 640,000 \text{ revolutions for Earth} \\ 1,300 \text{ revolutions for Saturn} \end{cases} \quad (92)$$

The orbital period of such a satellite is 4 hours for Earth and 12 hours for Saturn, giving

$$P \geq \begin{cases} 290 \text{ years for Earth} \\ 1.8 \text{ years for Saturn} \end{cases} \quad (93)$$

The quantities  $\Delta I$  and  $\Delta q$  are the maximum possible widths of the oscillations in inclination angle,  $I$ , and pericenter distance,  $q$ . The modulus,  $k$ , is simply a scale factor relating actual to maximum possible widths. The amplitude,  $\Delta \omega$ , of the oscillation in the argument of pericenter,  $\omega$ , is

$$\Delta \omega = \arcsin k$$

Hence, pericenter oscillates about the node and never reaches the points of maximum declination, north or south.

Since the quantities  $d\omega/dv$ ,  $I$ , and  $q$  are identical functions of  $\vartheta$  (and hence of  $\omega$ ), except for scale and origin, the phase-plane plot of figure 2 represents all three, the positive direction of the axis of ordinates being upward for  $d\omega/dv$  and downward for  $I$  and  $q$ . At the origin  $d\omega/dv=0$ ,  $I=I_c=I_e$ , and  $q=q_c=q_0+k\Delta q$ . Writing

$$r_p = qR$$

where  $r_p$  is the actual pericenter distance, and  $R$  is the equatorial radius of the planet, gives the "half-width" of the libration region:

$$\left. \begin{aligned} \Delta I &= \frac{2e_0 \sqrt{6J}}{q_0(1+e_0)} \cdot 0^\circ.955 \\ \Delta r_p &= \frac{2e_0 \sqrt{6J}}{1+e_0} \frac{R}{15} \end{aligned} \right\} \quad (94)$$

Note that  $\Delta I$  does not depend on the actual size of either the planet or the orbit, but only on the size of the orbit relative to the planet ( $q_0$ ); on the other hand,  $\Delta r_p$  depends only on the size of the planet ( $R$ ). Of course, they both depend on the orbital eccentricity in the same way. Numerical values for Earth and Saturn are

$$q_0 \Delta I = \frac{2e_0}{1+e_0} \begin{cases} 5'.6 \text{ for Earth} \\ 44' \text{ for Saturn} \end{cases} \quad (95)$$

$$\Delta r_p = \frac{2e_0}{1+e_0} \begin{cases} 42 \text{ km, Earth} \\ 3000 \text{ km, Saturn} \end{cases} \quad (96)$$

The maximum rates of change are

$$\left. \begin{aligned} \omega &: \begin{cases} 2''/\text{revolution, Earth} \\ 17''/\text{revolution, Saturn} \end{cases} \\ I &: \begin{cases} 0''.0014/\text{revolution, Earth} \\ 5''.4/\text{revolution, Saturn} \end{cases} \\ r_p &: \begin{cases} 16 \text{ cm/revolution, Earth} \\ 5 \text{ km/revolution, Saturn} \end{cases} \end{aligned} \right\} \quad (97)$$

#### THE SEPARATRIX

When the modulus,  $k$ , is equal to unity, the elliptic functions become hyperbolic functions (ref. 10):

$$\left. \begin{aligned} \operatorname{sn} \vartheta &= \tanh \vartheta \\ \operatorname{cn} \vartheta &= \operatorname{dn} \vartheta = \operatorname{sech} \vartheta \end{aligned} \right\} \quad (98)$$

and equations (88) of the preceding section become

$$\left. \begin{aligned} \psi &= v \sqrt{2b} \\ \vartheta &= \psi + \frac{1}{3} \gamma \arcsin (\tanh \psi) \\ \sin \omega &= \tanh \vartheta, \cos \omega = \operatorname{sech} \vartheta \\ \frac{d\omega}{dv} &= \sqrt{2b} \cos \omega \\ I &= I_c - \Delta I \cos \omega \\ \frac{dI}{dv} &= \frac{1}{2} \Delta I \sqrt{2b} \sin 2\omega \\ q &= q_0 + \Delta q (1 - \cos \omega) \\ \frac{dq}{dv} &= \frac{1}{2} \Delta q \sqrt{2b} \sin 2\omega \end{aligned} \right\} \quad (99)$$

and, of course, the parameters are still given by equations (89). The equations were derived under the assumptions that  $\omega_0=0$ ,  $\omega'_0 \geq 0$ . If  $\omega_0=\pi$ , replace  $\omega$  by  $\omega-\pi$ ; if  $\omega'_0 < 0$ , change the signs of  $d\omega/dr$  and  $\gamma$ .

The periodic character of the motion has disappeared, or, stated differently, the period has become infinite. The motion is aperiodic, or asymptotic. For large values of  $\vartheta$  the hyperbolic functions can be approximated by exponential functions:

$$\sin \omega = \tanh \vartheta \cong 1 - 2e^{-2\vartheta}$$

$$\cos \omega = \operatorname{sech} \vartheta \cong 2e^{-\vartheta}$$

The limiting values, as  $\vartheta$  approaches infinity, are

$$\left. \begin{aligned} \omega_\infty &= \frac{\pi}{2} \operatorname{sign}(\omega'_0) \\ I_\infty &= I_c \\ q_\infty &= q_0 + \Delta q \end{aligned} \right\} \quad (100)$$

The time required for  $\omega - \omega_\infty$ ,  $I - I_\infty$ , or  $q - q_\infty$  to decay by the factor 1/2 is

$$t = \frac{\ln 2}{\sqrt{2b}}$$

The number of orbital revolutions in this "decay time" is

$$N = \frac{0.7}{2\pi\sqrt{2b}} = 5.5 \frac{q_0^3(1+e_0)^3}{8e_0 J \sqrt{J}} \quad (101)$$

and this is a minimum when  $e_0=1/2$ ,  $q_0=1$ . Hence,

$$N \geq \left\{ \begin{array}{l} 72,000 \text{ revolutions for Earth} \\ 146 \text{ revolutions for Saturn} \end{array} \right\} \quad (102)$$

Recalling that the orbital period of such a satellite is 4 and 12 hours for Earth and Saturn, respectively, gives the decay time

$$t \geq \left\{ \begin{array}{l} 33 \text{ years for Earth} \\ 0.2 \text{ year for Saturn} \end{array} \right\} \quad (103)$$

Every case ( $\omega_0=0, \pi$ ;  $\omega'_0 > 0, < 0$ ) can be described in general terms as follows. If the orbital plane is initially on the equatorial side of the critical position, the apsidal motion is direct, otherwise retrograde. In every case pericenter asymptotically approaches the point of

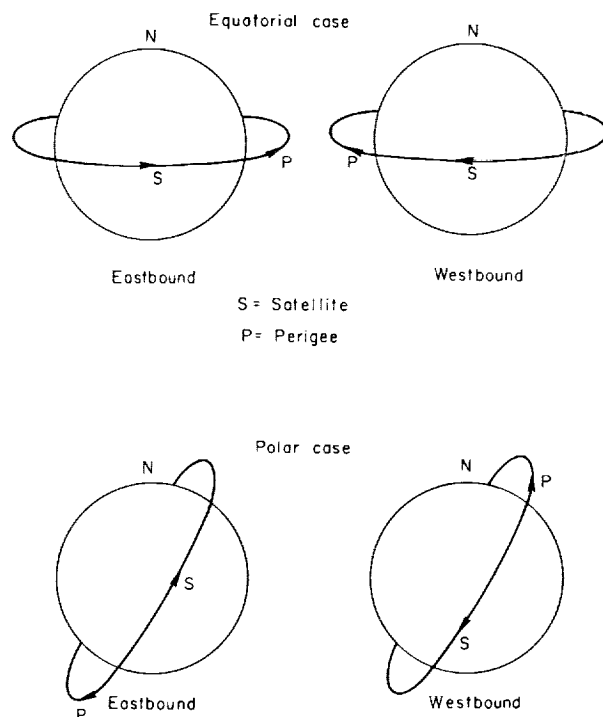


FIGURE 9.—Relative directions of motion of satellite and perigee.

maximum declination. This is shown schematically in figure 9.

#### THE TRANSITION REGION, $k > 1$

The region under consideration here is that portion of the elliptic region (outside the separatrix) in which the elliptic series converge more rapidly than the trigonometric ones. The transition from aperiodic motion to secular motion takes place in this region, with a concomitant decrease in the amplitudes of the long-period oscillations.

Equation (68) defines the transition region:

$$\left. \begin{aligned} k &> 1 \\ k^3 - k &\leq \frac{1.3}{\sqrt{J}} \frac{p_0}{2e_0} \end{aligned} \right\} \quad (104)$$

The specification of the region in terms of inclination angle will be given later.

The elliptic functions change their character when the modulus,  $k$ , is greater than unity, but their original character can be restored by means of the reciprocal modulus transformation (ref. 10, eq. 162.01). Thus, throughout this section the modulus of all elliptic functions and integrals will

be the reciprocal modulus,  $k_1$ :

$$\left. \begin{aligned} k_1 &= \frac{1}{k} < 1 \\ k_1' &= \sqrt{1 - k_1^2} \end{aligned} \right\} \quad (105)$$

Equations (74) to (83) now become

$$\left. \begin{aligned} \psi &= v\sqrt{2b} \operatorname{sign}(\omega_0') \\ \omega &= am \, k\vartheta = \operatorname{arc} \sin \operatorname{sn} k\vartheta \\ \sin \omega &= \operatorname{sn} k\vartheta \\ \cos \omega &= \operatorname{cn} k\vartheta \\ w &= k\gamma \operatorname{dn} k\vartheta \end{aligned} \right\} \quad (106)$$

with

$$\left. \begin{aligned} \vartheta &= \psi + \sum_{n=1}^{\infty} J_n(\psi) \gamma^n \\ J_1(\psi) &= \frac{1}{3} I_1(\psi) \\ J_2(\psi) &= \frac{1}{9} k(\operatorname{dn} k\psi) I_1(\psi) - \frac{1}{12} I_2(\psi) \\ J_3(\psi) &= -\frac{1}{54} \operatorname{sn} k\psi (\operatorname{cn} k\psi) I_1^2(\psi) \\ &\quad + \frac{1}{108} k^2 (\operatorname{dn}^2 k\psi) I_1(\psi) \\ &\quad - \frac{1}{36} k(\operatorname{dn} k\psi) I_2(\psi) + \frac{2}{135} I_3(\psi) \end{aligned} \right\} \quad (107)$$

and

$$I_n(\psi) = \frac{1}{k} \int_0^{k\psi} (k \operatorname{dn} \psi_1)^n d\psi_1 \quad (108)$$

The recurrence relations of reference 10, page 194, give

$$\left. \begin{aligned} I_1(\psi) &= am \, k\psi \\ I_2(\psi) &= kZ(k\psi) + \frac{E}{K} k^2\psi \\ I_3(\psi) &= \left(k^2 - \frac{1}{2}\right) am \, k\psi + \frac{1}{2} \operatorname{sn} k\psi \operatorname{cn} k\psi \end{aligned} \right\} \quad (109)$$

Equations (85) to (87) for pericenter distance, semilatus rectum, and inclination angle remain unchanged.

Equations (69) and (106) give the bounds

$$|w| \leq |k\gamma| \leq \begin{cases} 0.36 & \text{for Earth} \\ 1.60 & \text{for Saturn} \end{cases}$$

for the worst case of grazing, parabolic orbits. Hence, the truncated series

$$z = w + \frac{1}{3} w^2 \quad (110)$$

is in error by less than 1 percent for Earth and 17 percent for Saturn, and the error decreases with eccentricity. This approximation will be used throughout this section; also the parameters  $a$ ,  $b$ ,  $\gamma$ ,  $\beta$  will be approximated by their values at  $I_0 = \operatorname{arc} \tan 2$ :

$$\left. \begin{aligned} a &= -\frac{67}{300} j_0^2 e_0^2 \\ b &= \frac{1}{75} j_0^3 e_0^2 \\ \gamma &= -2.75 \sqrt{J} \frac{2e_0}{p_0} \operatorname{sign}(\omega_0') \\ \beta &= -\frac{4}{67} \end{aligned} \right\} \quad (111)$$

which are consistent in accuracy with equation (110).

The initial inclination angle,  $I_0$ , can now be expressed in terms of  $k$  as follows. Equations (30) to (33), with (111), give

$$5 \cos^2 I_0 - 1 = -\frac{4z_0}{67}$$

and this can be transformed, by equations (106) and (110), into

$$I_0 = \operatorname{arc} \tan 2 + 0^\circ.855 k\gamma \left(1 + \frac{1}{3} k\gamma\right) \quad (112)$$

Hence the width of this region, in terms of the range of permissible values of  $I_0$ , is

$$\Delta I_0 = 1^\circ.71 k\gamma \quad (113)$$

where  $k$  is the root of equation (68).

By equation (70),

$$\Delta I_0 \sim \sqrt[3]{J \frac{e_0^2}{q_0^2(1+e_0)^2}} \quad (114)$$

The range,  $\Delta I_0$ , is plotted versus the eccentricity,  $e_0$ , in figure 10, for grazing orbits ( $q_0 = 1$ ). The values in the figure can be scaled by dividing by  $q_0^{2/3}$ .

It is now convenient to regard  $J$ ,  $k$ ,  $e_0$ , and  $q_0$  as the primary parameters, since then  $I_0$  can easily

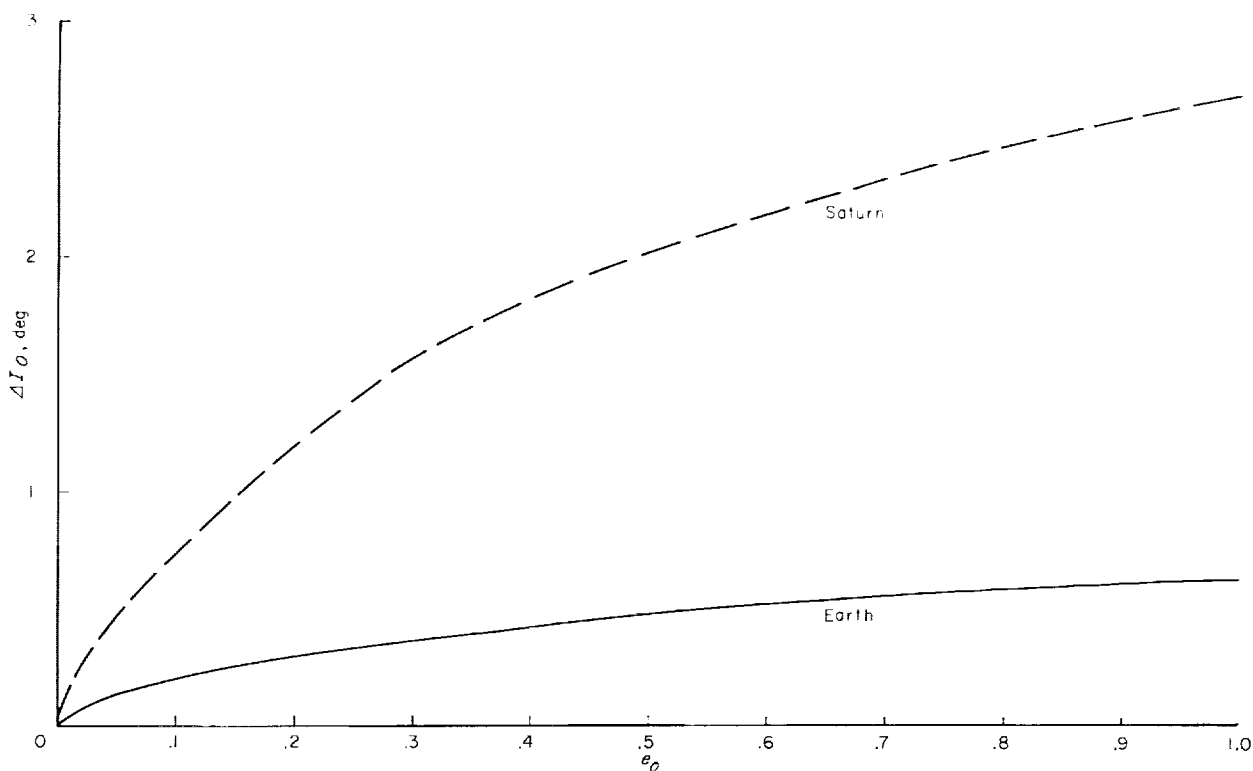


FIGURE 10.—Width of transition region versus eccentricity.

be obtained from equation (112). The well-known properties of the elliptic functions can then be used to express the solution in the following (approximate) form:

$$\left. \begin{aligned}
 \psi &= c\sqrt{2b} \operatorname{sign}(\omega_0') \\
 \vartheta &= \left(1 + \frac{\pi k \gamma}{6K_1}\right) \psi + \tilde{\vartheta} \\
 \omega &= \frac{\pi k}{2K_1} \vartheta + \tilde{\omega} \\
 \tilde{\vartheta} &= \frac{1}{3} \gamma \tilde{a} \tilde{m} k \psi \\
 \tilde{\omega} &= \tilde{a} \tilde{m} k \vartheta \\
 \xi &= \frac{4k\gamma}{67} (1 - dn k \vartheta) \left[ 1 - \frac{1}{6} k \gamma (1 + dn k \vartheta) \right] \\
 I &= I_0 + \tilde{I} \\
 q &= q_0 + \tilde{q} \\
 \tilde{I} &= -\frac{1}{4} \xi \left( 1 - \frac{3}{20} \xi \right) \\
 \tilde{q} &= -q_0 \xi (1 - \xi)
 \end{aligned} \right\} \quad (115)$$

where the tilde ( $\sim$ ) is used to denote periodic components, and the elliptic integral  $K_1 = K(k_1)$  is written with a subscript as a reminder that the modulus is  $k_1$ .

The common period of  $\tilde{\omega}$ ,  $\tilde{I}$ , and  $\tilde{q}$  is

$$P = \frac{k_1 K_1}{\pi \sqrt{2b} \left( 1 + \frac{\pi k \gamma}{6K_1} \right)} \text{ orbital revolutions} \quad (116)$$

and the secular velocity of pericenter is

$$\omega'_{\text{sec}} = \frac{\operatorname{sign} \omega_0'}{2P} \quad (117)$$

revolutions per orbital revolution of the satellite.

At the separatrix  $P = \infty$ ,  $\omega'_{\text{sec}} = 0$ , and the motion is aperiodic. As  $k$  increases,  $P$  decreases and  $\omega'_{\text{sec}}$  increases monotonically. Their values at the boundary of the transition region are shown in table I for Earth and Saturn, for grazing orbits ( $q_0 = 1$ ).

The elliptic functions  $\tilde{a} \tilde{m} u$  and  $dn u$  are shown in figures 11 and 12 in normalized form. For small values of the modulus,  $k_1$ , they can be represented by the approximations

$$\left. \begin{aligned} \tilde{a}m u &\cong \frac{1}{8} k_1^2 \sin \frac{\pi u}{K_1} \\ 1 - dn u &\cong \frac{1}{4} k_1^2 \left( 1 - \cos \frac{\pi u}{K_1} \right) \end{aligned} \right\} \quad (118)$$

As the modulus,  $k_1$ , increases, the curves are progressively distorted; near the separatrix the asymptotic approximations are

$$\left. \begin{aligned} \tilde{a}m u &\cong \frac{\pi}{2} - \operatorname{arcsin} \operatorname{sech} u - \frac{\pi u}{2K_1} \\ 1 - dn u &\cong 1 - \operatorname{sech} u \end{aligned} \right\} \quad (119)$$

Since  $\tilde{a}m u$  is an odd function and  $dn u$  is an even function, both with period  $2K_1$ , only half a period is shown in the figures; also the approximations (119) are only valid for  $|u| \leq K_1$ .

Figure 11 or equations (118) and (119) can be used directly to obtain the amplitude of the oscillatory functions  $\tilde{\omega}$  and  $\tilde{\vartheta}$ . For example, in the worst case of parabolic orbits, at the transition boundary

$$\max |\tilde{\omega}| = \begin{cases} 0^\circ.66, & \text{Earth} \\ 2^\circ.2, & \text{Saturn} \end{cases} \quad (120)$$

and

$$\max |\tilde{\vartheta}| = \begin{cases} 0^\circ.024, & \text{Earth} \\ 0^\circ.42, & \text{Saturn} \end{cases} \quad (121)$$

In other words, at the transition boundary the secular terms are overwhelming, and

$$\omega = \omega'_{\text{sec}} \nu \quad (122)$$

The inclination angle and pericenter distance can be studied by means of equations (115), (118), (119), figure 12, and table I. Their oscillatory components,  $\tilde{I}$  and  $\tilde{q}$ , like  $\tilde{\omega}$ , decrease monotonically with  $k_1$ . Near the transition boundary they can be represented by the approximations

$$\left. \begin{aligned} \xi &\cong \frac{1}{67} k_1 \gamma \left( 1 - \frac{1}{3} k \gamma \right) \left( 1 - \cos \frac{\pi k \vartheta}{K_1} \right) \\ \tilde{I} &= -\frac{1}{4} \xi \\ r_p &= r_{p,0} + \Delta r_p \\ \Delta r_p &= -r_{p,0} \xi \end{aligned} \right\} \quad (123)$$

Figures 13 through 16 show the values at the transition boundary, for grazing orbits, as a

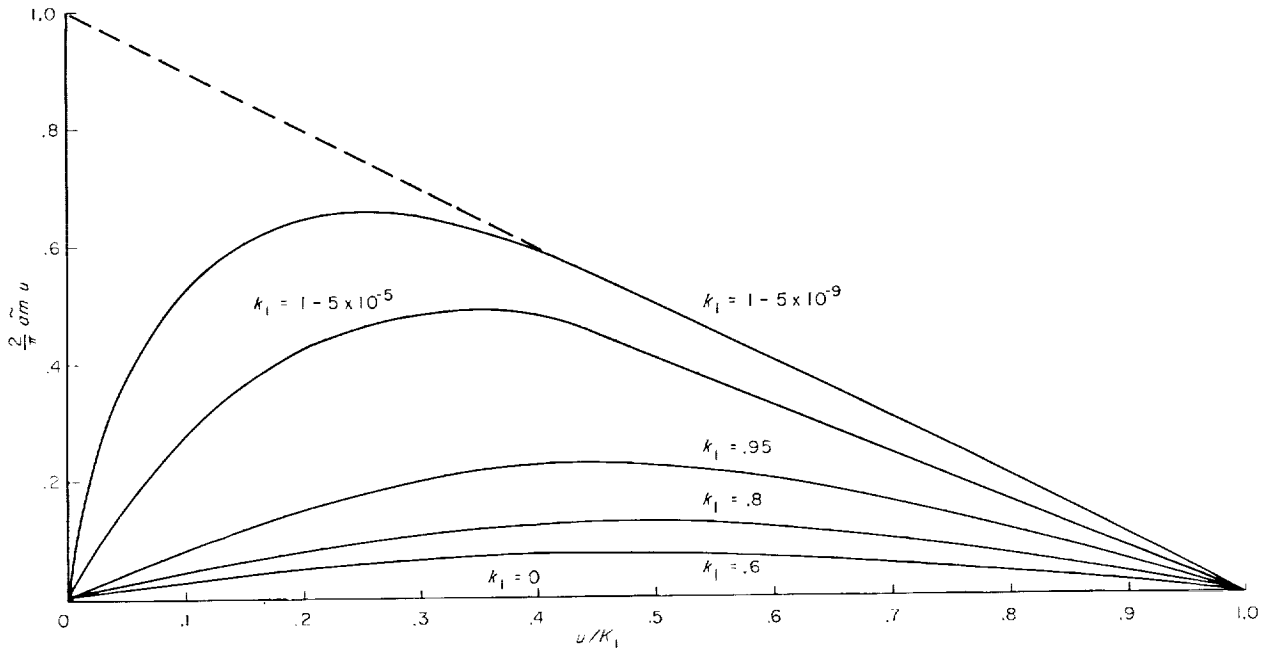
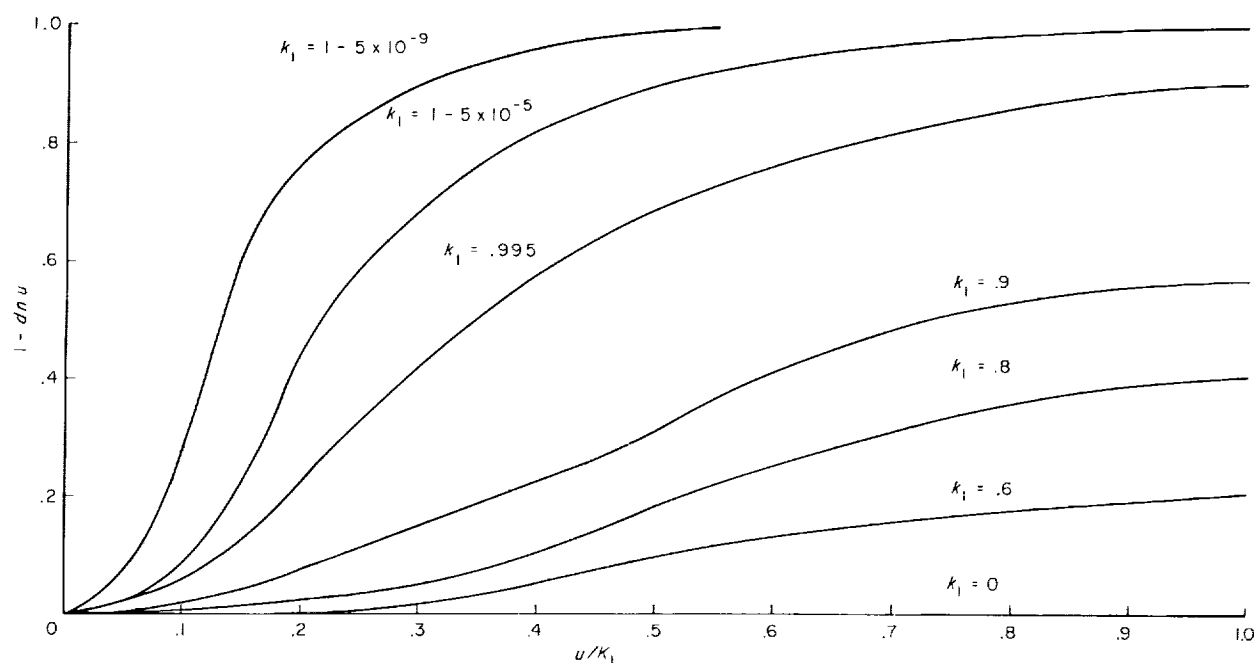


FIGURE 11.—The oscillatory part of  $am u$ .

FIGURE 12.-The elliptic function  $dn u$ .

function of eccentricity. For comparison, the values at the separatrix are also shown (see eqs. (94) to (96)).

Finally the approximation

$$\frac{d\omega}{dv} = k \sqrt{2b} \operatorname{sign} \omega_0' dn k \vartheta \quad (124)$$

shows that  $d\omega/dv$ ,  $I$ , and  $q$  exhibit the same behavior as functions of  $\vartheta$ , and again the phase-plane diagram of figure 2 represents all three functions with suitable choice of origin and scale.

#### SOLUTION OF THE APSIDAL EQUATIONS IN THE TRIGONOMETRIC REGION

The phase-plane integral in the trigonometric region is given by equations (72), which can be put in the form

$$\left. \begin{aligned} \omega_0' \frac{dv}{d\omega} &= 1 + \sum_{n=1}^{\infty} (1+z_0)(n+1) \zeta_{n+1} \lambda^n \\ \lambda &= \frac{-2a}{\omega_0'} \sin^2 \omega \end{aligned} \right\} \quad (125)$$

where the initial position of pericenter has been taken at the node:  $\omega_0 = 0$ . The powers of  $\lambda$  can be expressed as trigonometric polynomials, and then  $dv/d\omega$  takes the form of a Fourier cosine series:

$$\omega_0' \frac{dv}{d\omega} = \sum_{k=0}^{\infty} A_k \cos 2k\omega \quad (126)$$

The recurrence relations for the  $\zeta_n$  (eqs. (55)) lead to the following equations for the  $A_k$ :

$$\left. \begin{aligned} A_0 &= 1 + (b + 2a\omega_0') \sum_{n=1}^{\infty} \binom{2n}{n} \frac{B_n}{(2\omega_0')^{2n}} \\ A_k &= 2(-1)^k (b + 2a\omega_0') \sum_{n=k}^{\infty} \binom{2n}{n-k} \frac{B_n}{(2\omega_0')^{2n}} \end{aligned} \right\} \quad (127)$$

where

$$\left. \begin{aligned} B_0 &= \frac{1}{b} \\ B_{n+1} &= \frac{-2a\omega_0'}{n+1} B_n + b(b + 2a\omega_0') \sum_{k=0}^n \frac{B_k B_{n-k}}{n+1-k} \end{aligned} \right\} \quad (128)$$

The first few are

$$B_1 = 1$$

$$B_2 = \frac{3}{2} b + 2a\omega_0'$$

$$B_3 = \frac{5}{2} b^2 + \frac{20}{3} ba\omega_0' + (2a\omega_0')^2$$

$$B_4 = \frac{35}{8} b^3 + \frac{35}{2} b^2 a\omega_0' + \frac{65}{3} b(a\omega_0')^2 + (2a\omega_0')^3$$

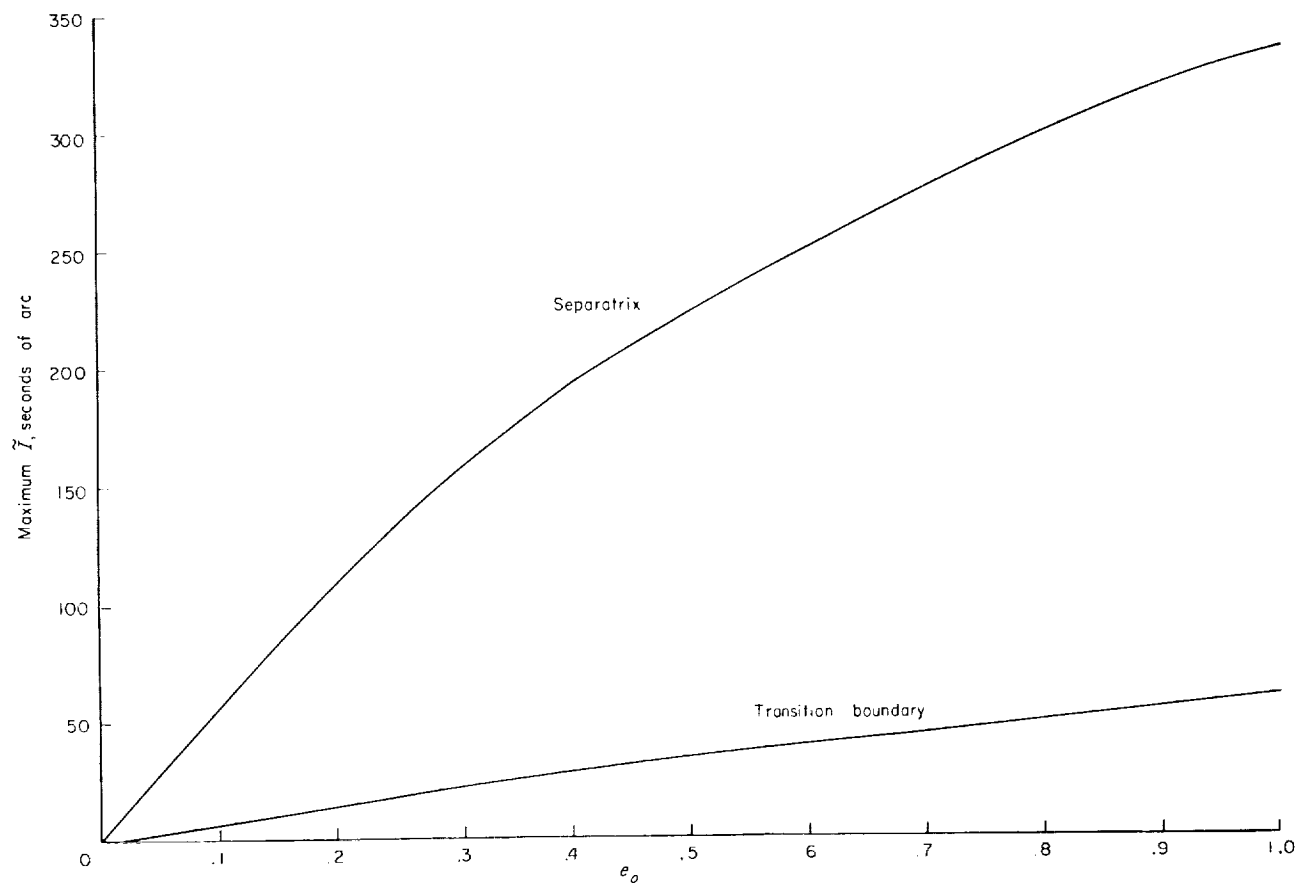


FIGURE 13.—Variation of inclination angle in the transition region for Earth.

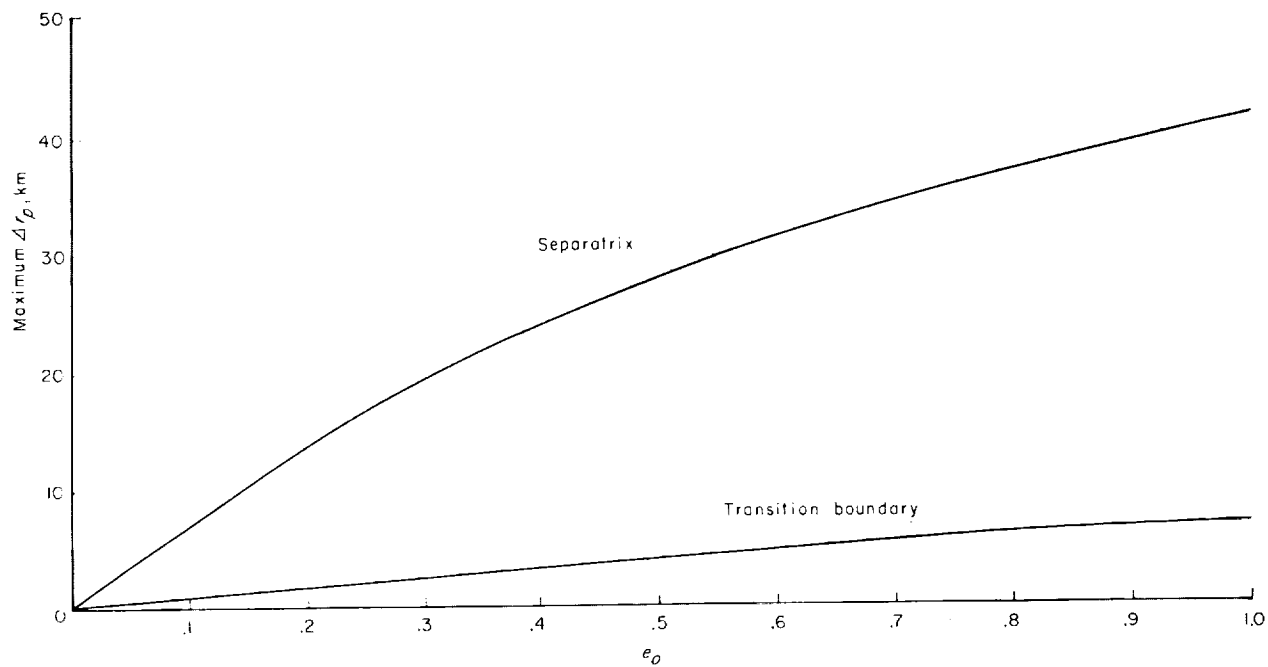


FIGURE 14.—Variation of perigee distance in the transition region for Earth.

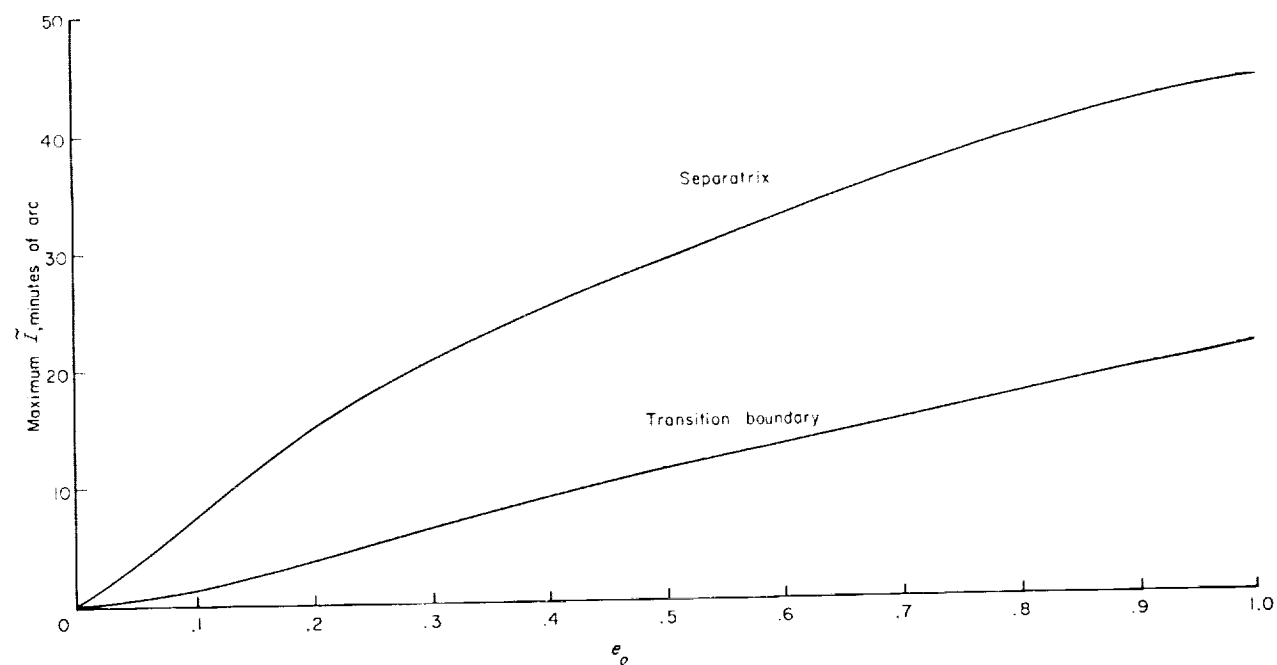


FIGURE 15.—Variation of inclination angle in the transition region for Saturn.

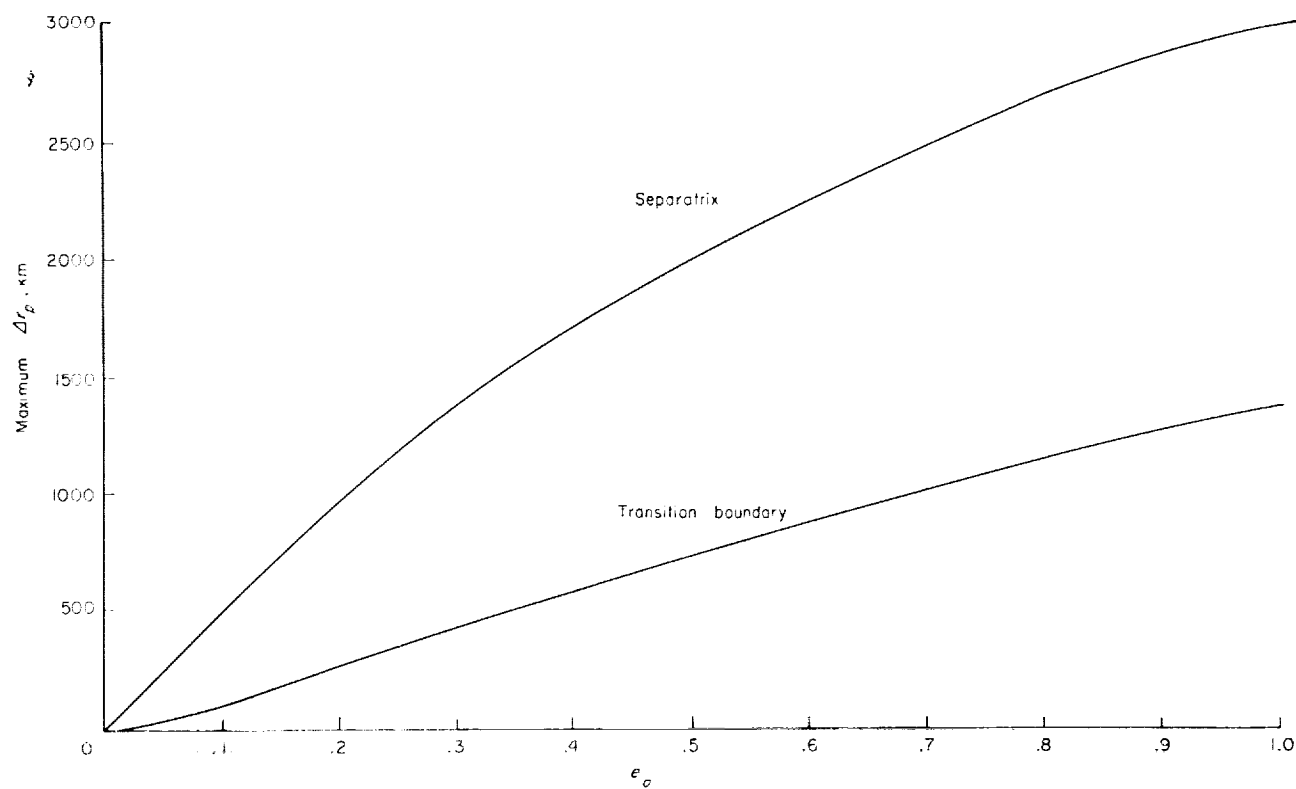


FIGURE 16.—Variation of pericenter distance in the transition region for Saturn.

Equation (126) can be integrated to give  $v$  as a Fourier series in  $\omega$ , which in turn can be inverted to give  $\omega$  as a Fourier series in  $v$  (ref. 7, appendix E, eqs. (E25)):

$$\omega = \omega_s' v + \sum_{n=1}^{\infty} \omega_n \sin 2n\omega_s' v \quad (129)$$

where the secular velocity is

$$\omega_s' = \frac{\omega_0'}{A_0} \quad (130)$$

and the coefficients of the harmonics are infinite series, the leading terms being

$$\left. \begin{aligned} \omega_1 &= -\frac{1}{2} \frac{A_1}{A_0} - \frac{A_1 A_2}{A_0^2} + \frac{1}{16} \frac{A_1^3}{A_0^3} \dots \\ \omega_2 &= -\frac{1}{4} \frac{A_2}{A_0} + \frac{1}{4} \frac{A_1^2}{A_0^2} \dots \\ \omega_3 &= -\frac{1}{6} \frac{A_3}{A_0} + \frac{3}{8} \frac{A_1 A_2}{A_0^2} - \frac{3}{16} \frac{A_1^3}{A_0^3} \dots \end{aligned} \right\} \quad (131)$$

The variable  $x$  can be obtained from equation (31)

$$\frac{dx}{dv} = -b \sin 2\omega \quad (31)$$

by expanding  $\sin 2\omega$  in a Fourier series in  $v$ :

$$\sin 2\omega = \sum_{n=1}^{\infty} S_n \sin 2n\omega_s' v \quad (132)$$

where

$$\left. \begin{aligned} S_1 &= 1 + \omega_2 - \frac{3}{2} \omega_1^2 \dots \\ S_2 &= \omega_1 + \omega_3 - 2\omega_1\omega_2 - \frac{1}{3} \omega_1^3 \dots \\ S_3 &= \omega_2 + \frac{1}{2} \omega_1^2 \dots \\ S_4 &= \omega_3 + \omega_1\omega_2 + \frac{1}{6} \omega_1^3 \dots \end{aligned} \right\} \quad (133)$$

Substituting in equation (31) and integrating gives

$$x - x_0 = -b \sum_{n=1}^{\infty} \frac{S_n}{2n\omega_s'} (1 - \cos 2n\omega_s' v)$$

and then equation (30) gives  $\xi$ :

$$\xi = \sum_{n=1}^{\infty} \xi_n (1 - \cos 2n\omega_s' v) \quad (134)$$

where

$$\xi_n = \frac{j_0^2 e_0^2 (3e_0^2 - 1) (1 - e_0^2) S_n}{24n\omega_s'} \quad (135)$$

and then the inclination angle and pericenter distance are given by equation (87)

$$\left. \begin{aligned} q &= \frac{q_0}{1 + \xi} \\ \cos^2 I &= \cos^2 I_0 (1 + \xi) \end{aligned} \right\} \quad (136)$$

These results can be summarized by retaining, for simplicity, only terms through the second order in  $J$ :

$$\left. \begin{aligned} \omega &= \omega_s' v + \omega_1 \sin 2\omega_s' v + \omega_2 \sin 4\omega_s' v \\ \xi &= \xi_1 (1 - \cos 2\omega_s' v) + \xi_2 (1 - \cos 4\omega_s' v) \end{aligned} \right\} \quad (137)$$

where

$$\omega_s' = \omega_0' \left[ 1 - 2 \frac{b + 2a\omega_0'}{(2\omega_0')^2} \right] \quad (138)$$

$$\left. \begin{aligned} \omega_1 &= \frac{b + 2a\omega_0'}{(2\omega_0')^2} \left[ 1 + 4 \frac{b + a\omega_0'}{(2\omega_0')^2} \right] \\ \omega_2 &= \frac{(b + 2a\omega_0')(b + 4a\omega_0')}{4(2\omega_0')^4} \\ \xi_1 &= \frac{j_0^2 e_0^2 (3e_0^2 - 1) (1 - e_0^2)}{24\omega_0'} \left[ 1 + 2 \frac{b + 2a\omega_0'}{(2\omega_0')^2} \right] \\ \xi_2 &= \frac{j_0^2 e_0^2 (3e_0^2 - 1) (1 - e_0^2) (b + 2a\omega_0')}{24(2\omega_0')^3} \end{aligned} \right\} \quad (139)$$

Thus the apsidal motion consists of a secular term and long-period terms, while the inclination angle and pericenter distance contain only long-period terms. The secular velocity,  $\omega_s'$ , is a monotonic function of initial inclination angle,  $I_0$ , reaching a maximum for equatorial orbits and a minimum for polar orbits:

$$\omega_s' \cong \frac{2J}{p_0^2}, \text{ equatorial}$$

$$\omega_s' \cong -\frac{J}{2p_0^2}, \text{ polar}$$

Thus, for equatorial orbits,

$$\omega_s' \leq \begin{cases} 1^\circ.15/\text{revolution, Earth} \\ 72^\circ/\text{revolution, Saturn} \end{cases}$$

and the corresponding period of the long-period oscillations is

$$P = \begin{cases} 156 \text{ orbital revolutions, Earth} \\ 2.5 \text{ orbital revolutions, Saturn} \end{cases}$$

It may also be noted that the secular velocity of pericenter is essentially independent of the orbital eccentricity,  $e_0$ , while the amplitudes of all the long-period oscillations contain  $e_0^2$  as a factor.

There are certain values of  $I_0$  that lead to simple exact solutions, namely those for which the coefficient  $\omega_n$  or  $\xi_n$  vanishes.

First, if the inclination angle,  $I_0$ , has either of the values

$$I_0 = \begin{cases} 62^\circ.6 \\ 72^\circ.4 \end{cases} \quad (140)$$

then  $b + 2a\omega'_0 = 0$  and the exact solution is

$$\left. \begin{aligned} \omega &= \omega'_0 r \\ \xi &= \frac{j_0^2 e_0^2 (3e_0^2 - 1)(1 - e_0^2)}{24\omega'_0} (1 - \cos 2\omega'_0 r) \end{aligned} \right\} \quad (141)$$

that is, the apsidal motion is purely secular.

On the other hand, if  $I_0$  has either of the values

$$I_0 = \begin{cases} 0^\circ \\ 54^\circ.7 \end{cases} \quad (142)$$

then all the  $\xi_n$  vanish, so that  $I$  and  $q$  are constant. In this case the apsidal equations (31) become

$$\frac{d\omega}{dr} = x_0 + a \cos 2\omega$$

and the exact solution is

$$\left. \begin{aligned} \tan \omega &= \sqrt{\frac{x_0 + a}{x_0 - a}} \tan (r \sqrt{x_0^2 - a^2}) \\ I &= I_0 \\ q &= q_0 \end{aligned} \right\} \quad (143)$$

This is essentially the same as the equation relating the true and eccentric anomalies in Keplerian motion (ref. 11, pp. 62-63). The series solution for  $\omega$  is (129), where now the coefficients are, in closed form,

$$\omega'_s = \sqrt{x_0^2 - a^2} \quad (144)$$

$$\omega_n = \frac{1}{n} \left( \frac{a}{x_0 + \sqrt{x_0^2 - a^2}} \right) \quad (145)$$

It may be noted that the phase-plane diagram of figure 2 cannot represent all three variables  $d\omega/dr$ ,  $I$ , and  $q$  in the trigonometric region, since their long-period components vanish at different inclination angles (eqs. (140) and (142)).

The amplitudes of the long-period perturbations decrease very rapidly with distance from the transition boundary. Thus, for example, when  $I_0 = 70^\circ$ ,

$$|I - I_0| \leq \begin{cases} 1''.7, \text{ Earth} \\ 1'.8, \text{ Saturn} \end{cases}$$

$$\Delta r_p \leq \begin{cases} 0.30 \text{ km, Earth} \\ 165 \text{ km, Saturn} \end{cases}$$

$$\omega_1 \leq \begin{cases} 13'', \text{ Earth} \\ 14', \text{ Saturn} \end{cases}$$

## DISCUSSION

A direct, analytic comparison of the various treatments of the critical inclination problem is almost impossible because of the multiplicity of notations, approximations, and starting points. No one theory has a monopoly on either simplicity or accuracy. The present theory has the virtue of using an intermediate orbit that eliminates the need of considering the variations in the eccentricity, but the price to be paid is the introduction of infinite series rather than closed-form solutions.

One novel feature of the present theory is the emergence of certain inclination angles that eliminate completely the long-period oscillations either in the apsidal motion or in the inclination angle and pericenter distance. The other is that the convergence of the infinite series is rigorously proved, so that precise error estimates are available.

## CONCLUDING REMARKS

The present theory presents solutions of the satellite orbit problem that do not exhibit singularities at the critical inclination angle. Series representations are obtained, their regions of convergence are exhibited, and quantitative measures of their speeds of convergence are provided for use in numerical computations.

Essentially similar results have been obtained by several authors. However, the development of new methods of solving old problems has always played an important role in the growth of any science. The present method can be used to

reduce any axially symmetric problem to two dimensions. Thus it can be applied, for example, to study the effect of the earth's magnetic field on the orbit of an electrically charged satellite, or to study the long-period and secular effects of the

sun and moon on the orbits of near earth satellites.

AMES RESEARCH CENTER

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TABLE I.—PARAMETERS AT THE BOUNDARY OF THE TRANSITION REGION

(a) Earth						
Eccentricity	$k_1$	$K_1$	$ \gamma $	$\sqrt{2b}$	Period (revolutions)	Secular motion of pericenter per revolution
0	0	1.571	0	0	$\infty$	0''
.001	.0394	1.571	.000220	$.104 \times 10^{-7}$	$2 \times 10^6$	0''.34
.01	.0846	1.574	.00218	$.101 \times 10^{-6}$	$4 \times 10^5$	1''.6
.1	.176	1.583	.0200	$.785 \times 10^{-6}$	$1 \times 10^5$	5''.9
.25	.227	1.592	.0440	$.134 \times 10^{-5}$	$8 \times 10^4$	8''.0
.5	.267	1.600	.0733	$.155 \times 10^{-5}$	$8 \times 10^4$	8''.0
.75	.289	1.605	.0943	$.146 \times 10^{-5}$	$9 \times 10^4$	7''.1
1	.303	1.609	.110	$.131 \times 10^{-5}$	$1 \times 10^5$	6''.1
(b) Saturn						
0	0	1.571	0	0	$\infty$	0'
.001	.0784	1.573	.00174	$.515 \times 10^{-5}$	7600	1'.4
.01	.167	1.582	.0171	$.501 \times 10^{-4}$	1600	6'.6
.1	.339	1.619	.145	$.388 \times 10^{-3}$	400	27'
.25	.429	1.652	.348	$.661 \times 10^{-3}$	270	40'
.5	.495	1.683	.580	$.800 \times 10^{-3}$	240	45'
.75	.532	1.704	.745	$.754 \times 10^{-3}$	270	40'
1	.552	1.717	.870	$.676 \times 10^{-3}$	300	36'

